# Graphs with Large Clique Number whose Second Largest Eigenvalue does not Exceed $(\sqrt{5}-1)/2$

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**Abstract**

In 1993, Cao and Hong [J. Graph Theory, 17 (1993), 325–331] posed the problem of characterizing graphs whose second largest eigenvalue is less than the golden section bound. In further considerations, the problem is extended to 'less than or equal to the golden section'. Several results giving partial characterizations appeared in the proceeding years, and what have remained are the most complicated cases. These cases are treated very sporadically in the period of the next 25 years. In this paper, we give a positive resolution to the problem for graphs containing a large clique. Actually, we characterize graphs whose second largest eigenvalue does not exceed the golden section bound and whose clique number is at least 54. If a graph has a pendant vertex, the result is improved to clique number at least 8.

**Mathematics Subject Classifications:** 05C50, 05C22

## 1 Introduction

In this paper we deal with finite undirected graphs without loops or multiple edges. For such a graph $G$, we denote its vertex set by $V(G)$ and edge set by $E(G)$. The number of vertices is denoted by $n$ and called the *order* of $G$. We use $H \subset G$ to designate that $H$ is an induced subgraph of $G$. In particular, if $H$ is a complete graph, then it is called a *clique* (of $G$). The *clique number* $\omega$ (or $\omega(G)$) is the order of a maximum clique of $G$.

A vertex of degree 1 is referred to as a *pendant vertex*. As usual, $K_n$, $P_n$ and $C_n$ denote the complete graph, the path and the cycle of order $n$, respectively. Also, $K_{s,t}$ denotes the complete bipartite graph with $s$ vertices in one colour class and $t$ vertices in the other

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colour class. We write $W_{s,t}$ for the graph obtained by taking the complete graph $K_s$ with vertex set $\{u_1, u_2, \dots, u_s\}$ and $t$ ($t \leqslant s$) isolated vertices $v_1, v_2, \dots, v_t$, and inserting the edges $u_1v_1, u_2v_2, \dots, u_tv_t$. For a graph $G$, $sG$ and $\overline{G}$ denote the disjoint union of $s$ copies of $G$ and the complementary graph of $G$, respectively. For $G$ and $H$, $G \vee H$ denotes their *join*, i.e., the graph obtained by inserting an edge between every vertex of $G$ and every vertex of $H$. If $F$ is an induced subgraph of $G$, then $H \vee (F|G)$ is the graph obtained by inserting an edge between every vertex of $H$ and every vertex of $F$ (This operation reduces to the join when $F \cong G$.)

We write $\lambda_2$ to denote the second largest eigenvalue of $G$, that is the second largest eigenvalue of the standard $\{0,1\}$-adjacency matrix of $G$. Graphs whose second largest eigenvalue is comparatively small have received a great deal of attention in the last five decades. Many results obtained before 2015 are surveyed in [24], whereas for the recent progress we refer the reader to [5, 16, 17, 26]. In particular, graphs with $\lambda_2 \leqslant \frac{1}{3}$ are known for a long period, and graphs with $\lambda_2 \leqslant \frac{1}{2}$ have been determined in 2023 [26]. Since every rational eigenvalue is an integer, $\lambda_2$ never attains the previous bounds. Graphs with $\lambda_2 \leqslant 1$ are extensively studied, and some notable references are [7, 10, 20, 21]. According to [17], to conclude determination of these graphs it remains to identify those which contain an induced subgraph isomorphic to $K_4 - e$, that is the graph obtained by removing an edge from the complete graph of order 4.

Another upper bound for $\lambda_2$, located between the previous two, has been studied, again with the absence of a complete characterization of the corresponding graphs. This is the so-called *golden section* (also known as the *golden ratio*) $\frac{\sqrt{5}-1}{2} \approx 0.618$. This constant appears in the spectrum of many graphs, say it is the second largest eigenvalue of the 4-vertex path. In 1993, Cao and Hong formulated the following problem.

**Problem 1.** *[4]* Characterize graphs with $\frac{1}{3} < \lambda_2 < \frac{\sqrt{5}-1}{2}$.

Henceforth, we make the reading easier by denoting $\sigma = \frac{\sqrt{5}-1}{2}$ and abbreviating a connected graph with $\lambda_2 \leqslant \sigma$ to a *$\sigma$-graph.* This terminology is consistent with [2, 7, 9, 23]. Before this paper, it has been shown that every $\sigma$-graph is either complete multipartite, or an induced subgraph of the 5-vertex cycle, or contains a triangle and its diameter is at most 3 [8]. Further characterizations of those with a triangle are given in the same reference. Considering the existing results on $\sigma$-graphs, it is worth mentioning that Simić has proved in [23] that the set of minimal forbidden induced subgraphs for $\lambda_2 < \sigma$ is finite. A partial list of these subgraphs can be found in [9]. The same result is extended to $\sigma$-graphs by Cvetković and Simić in [8], again with the absence of the complete list of the corresponding forbidden subgraphs.

All former results concerning $\sigma$-graphs were noteworthy and, in contrast to them, a complete characterization of these graphs requires intensive technical considerations and a multiple case analysis. In the light of the aforementioned unresolved case for $\lambda_2 \leqslant 1$, a possible method to deal with the golden section bound is to determine $\sigma$-graphs that do not contain $K_4 - e$ as an induced subgraph (they are already identified among graphs with $\lambda_2 \leqslant 1$), and consider the remaining ones (with an induced subgraph isomorphic to $K_4 - e$). However, it occurs that the existence of such a subgraph is very common for graphs with small $\lambda_2$, and this approach would be rather difficult. In this contribution, our attention is focused on $\sigma$-graphs with a comparatively large clique number. This is motivated by the following empirical observations. By considering connected graphs whose second largest eigenvalue is bounded by some of the aforementioned constants, one may deduce that they have smaller diameter (see [23]), and consequently more 'round' shape followed by higher clique number and girth (length of a shortest cycle). On the contrary, for large values of $\lambda_2$, the graphs have larger diameter, and consequently more 'path-like' shape followed by smaller clique number and girth. An explanation in the case of regular graphs is offered by Cvetković [6], and Alon and Chung [1]. The latter result is the famous Expander Mixing Lemma that can also be found in [24], and whose concept appeared in the work of Haemers [14]. By observing that $\sigma$-graphs do not deviate from the previous structural description, one may see that a restriction on the clique number arises in a natural way.

To formulate our main result, we set $D_{a,b} \cong K_1 \vee (K_a \cup K_b)$, for $b \geqslant a \geqslant 2$, and

$$\mathcal{D} = \{H_s \vee (K_s|W_{s,s}) : s \geqslant 1\}, \text{ where } \overline{H}_s = s\big((K_1 \vee C_5) \cup \overline{W}_{3,3} \cup D_{2,3} \cup W_{s,s}\big).$$

The result reads as follows.

**Theorem 2.** *Let $G$ be a connected graph that satisfies either*

*(i)* $\omega(G) \geqslant 54$ *or*

*(ii)* $\omega(G) \geqslant 8$ *and $G$ has a pendant vertex.*

*Then $\lambda_2(G) \leqslant \frac{\sqrt{5}-1}{2}$ holds if and only if $G$ is an induced subgraph of some graph of $\mathcal{D} \cup \{\overline{D}_{2,4} \vee K_{52}\}$.*

Since $\omega(\overline{D}_{2,4} \vee K_{52}) = 54$, the previous theorem implies the following result.

**Corollary 3.** *For a connected graph $G$ with $\omega(G) \geqslant 55$, $\lambda_2(G) \leqslant \frac{\sqrt{5}-1}{2}$ holds if and only if $G$ is an induced subgraph of some graph of $\mathcal{D}$.*

The connectedness of $G$ is not a significant obstacle, since a disconnected graph with $\lambda_2 \leqslant \sigma$ is the disjoint union of a connected one and a set of isolated vertices, provided by $\lambda_2(2K_2) > \sigma$. The proof of Theorem 2 is based on two statements, each belonging to a wider context. The first of them is the forthcoming Theorem 5 concerning the determinant of a particular complex matrix and an application giving the upper bound for its second largest eigenvalue under certain additional assumptions. The second one (Theorem 7) establishes all connected graphs with $\lambda_2 \leqslant \sigma$ under the caveat that their clique number is at least 8 and they do not contain particular induced subgraphs.

Regarding a comparison with recently obtained graphs satisfying $\lambda_2 < \frac{1}{2}$, it is worth mentioning that these graphs are classified into the 13 structured types and each of them contains graphs obtained by combining unions and joins of complete graphs, or complete bipartite graphs, or their complements. Of course, all of them with large clique are covered by Theorem 2. Concerning the difference, i.e., graphs characterized by the same theorem but not belonging to any class with $\lambda_2 < \frac{1}{2}$, it occurs that such graphs are obtained easily. For instance, $\lambda_2(\overline{D}_{2,4} \vee K_t) > \frac{1}{2}$ holds whenever $t \geqslant 3$, and there is a similar conclusion for graphs of $\mathcal{D}$. The method applied in [26] is elegant and relies on facts that the complement of a non-trivial graph with $\lambda_2 > \frac{1}{2}$ must be disconnected with each component containing a dominating vertex. This approach is not applicable in our study, since in general neither of the previous restrictions holds for a complement of a $\sigma$-graph. However, we also deal with complementary graphs having some structural restrictions, where these restrictions refer to the discussion on the existence of prescribed components called special components (see Section 4). The limits on clique numbers given in Theorem 2 arise from the existence of these components, which can be seen from the proof of the necessity in this theorem. Another comparison with [26] is given in the last section of this paper.

Our contribution echoes the results of [23] since this reference classifies graphs with a relaxed restriction $\lambda_2 < \sigma$ into the nine structured types, and a simple analysis on them shows that the clique number of these graphs is $\leqslant 54$, unless they are complete multipartite. Accordingly, Theorem 2(i) starts with a boundary case $\omega = 54$ and continues with $\omega > 54$.

We proceed with other related results. A *cograph* is a graph which does not contain $P_4$ as an induced subgraph. There is no inclusion between the class of $\sigma$-graphs and the class of cographs; for example, a $\sigma$-graph of diameter 3 is not a cograph, whereas a cograph that contains $2K_2$ as an induced subgraph is not a $\sigma$-graph. However, there is an interplay between these classes, and in some investigations they are considered simultaneously. In particular, this occurs in the context of the polynomial reconstruction [2], spectral characterization [4, 11], control theory [13] or the rank of graphs [22].

There are results concerning relationships between the clique number and some other eigenvalues of a graph. We know from Nikiforov's [18, 19] that

$$\lambda_1 \leqslant \sqrt{\frac{\omega-1}{\omega}2m} < \frac{\omega-1}{\omega} \quad \text{and} \quad \lambda_n < -\frac{2^{\omega-1}m^{\omega}}{\omega n^{2\omega-1}},$$

where $\lambda_1$ is the largest eigenvalue, $\lambda_n$ is the least eigenvalue, $n$ is the order and $m$ is the number of edges; the latter inequality for $\lambda_1$ is the classical Wilf's inequality [25]. Since the clique number of graphs of Theorem 2 can be computed explicitly (for $\mathcal{D}$, in terms of $s$), the previous inequalities give upper bounds on $\lambda_1$ and $\lambda_n$ for the obtained $\sigma$-graphs. More relationships between $\lambda_1$ and $\omega$ are obtained by Bollobás and Nikiforov [3]. Another interplay between $\lambda_1, \lambda_n$ and $\omega$ is obtained by Gregory et al. [12]. Each of the foregoing inequalities can also be found in [24].

The remaining sections are organized in the following way. Section 2 is preparatory. In Section 3, we prove the sufficiency in Theorem 2. The necessity, which is the main contribution of this paper, is proved in Section 4. Some concluding remarks and comparisons are given in Section 5.

## 2 Preliminaries

This is a short section collecting all what is needed. We use $I, J, O$, $\mathbf{j}$ and $\mathbf{0}$ to denote the identity matrix, the all-1 matrix, the all-0 matrix, the all-1 vector and the all-0 vector, respectively. The size may be given in the subscript.

We write $N(v)$ and $d(v)$ for the neighbourhood and the degree of a vertex $v$ of a graph $G$. For any subset $U$ of $V(G)$ and vertex $v \notin U$, we denote by $N_U(v) = N_G(v) \cap U$ and $N_U[v] = N_U(v) \cup \{v\}$ the open and the closed neighbourhood of $v$ in $U$, respectively. To simplify, for an induced subgraph $H$ of $G$, $N_{V(H)}(v)$ is abbreviated to $N_H(v)$. The graph induced by the vertex set $U$ is denoted by $G[U]$, and $G\backslash U$ denotes the graph obtained by removing the vertices of $U$. A *component* of a graph is a maximal connected induced subgraph.

In the entire paper, we assume that the eigenvalues of a graph $G$ are indexed non-increasingly as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The corresponding characteristic polynomial is denoted by $\Phi(G)$. To avoid an unnecessary quoting of the order $n$, we also use $\lambda_{\min}$ to denote the least eigenvalue. The same notation is used for every real symmetric matrix.

We will use the next lemma.

**Lemma 4** ([15, p. 20])**.** *The following statements hold true:*

*(i) Let*

$$S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

*be a square block matrix, such that $M$ is invertible. Then* $\det(S) = \det(M)\det(Q - PM^{-1}N)$;

*(ii) If $A$ is an $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{y}$ are $n \times 1$ vectors, then* $\det(A + \mathbf{x}\mathbf{y}^\intercal) = \det(A) + \mathbf{y}^\intercal \operatorname{adj}(A)\mathbf{x}$, *where* $\operatorname{adj}(A)$ *stands for the adjugate matrix.*

Item (i) is known as the Schur's formula. The entire lemma holds for complex matrices and vectors.

## 3 Sufficiency

It follows from the Interlacing Theorem that the property $\lambda_2 \leqslant \sigma$ is hereditary in the sense that if it holds for a graph $G$ then it holds for every induced subgraph of $G$. Therefore, to prove the sufficiency in Theorem 2, it is enough to prove that the graphs of $\mathcal{D}$ and $\overline{D}_{2,4} \vee K_{52}$ are $\sigma$-graphs.

Observe that Lemma 4(ii), implies

$$\det(A) = \det(A + \mathbf{x}\mathbf{y}^\intercal - \mathbf{x}\mathbf{y}^\intercal) = \det(A + \mathbf{x}\mathbf{y}^\intercal) - \mathbf{y}^\intercal \operatorname{adj}(A + \mathbf{x}\mathbf{y}^\intercal)\mathbf{x}, \tag{1}$$

where, of course, $\mathbf{x}, \mathbf{y}$ and $A$ are of feasible size. We proceed with the following theorem.

**Theorem 5.** *Let*

$$A = \begin{pmatrix} A_1 & \alpha\beta^\intercal \\ \beta\alpha^\intercal & A_2 \end{pmatrix},$$

*where $\alpha$ and $\beta$ are real column vectors. Then*

$$\begin{aligned}\det(\lambda I - A) = &\det(\lambda I - A_1 + \alpha\alpha^\intercal)\det(\lambda I - A_2) + \det(\lambda I - A_1)\det(\lambda I - A_2 + \beta\beta^\intercal) \\ &- \det(\lambda I - A_1 + \alpha\alpha^\intercal)\det(\lambda I - A_2 + \beta\beta^\intercal). \end{aligned} \quad (2)$$

*Moreover, if $A$ is real, non-negative and symmetric, $\tau$ is a real number and*

$$\min\{\lambda_{\min}(\alpha\alpha^\intercal - I - A_1), \lambda_{\min}(\beta\beta^\intercal - I - A_2)\} \geqslant -1 - \tau,$$

*then $\lambda_2(A) \leqslant \tau$, with equality when $\lambda_1(A) > \tau$ and $\lambda_{\min}(\alpha\alpha^\intercal - I - A_1) = \lambda_{\min}(\beta\beta^\intercal - I - A_2) = -1 - \tau$.*

*Proof.* We set $\mathbf{r}^\intercal = (\alpha^\intercal, \beta^\intercal)$ and

$$B = \begin{pmatrix} \lambda I - A_1 + \alpha\alpha^\intercal & O \\ O & \lambda I - A_2 + \beta\beta^\intercal \end{pmatrix}.$$

If $B$ is invertible, then by (1) and Lemma 4, we have

$$\begin{aligned}\det(\lambda I - A) &= \det(B - \mathbf{r}\mathbf{r}^\intercal) = \det(B) - \mathbf{r}^\intercal \operatorname{adj}(B)\mathbf{r} \\ &= \det(B) - \mathbf{r}^\intercal \det(B)\ B^{-1}\mathbf{r} \\ &= \det(\lambda I - A_1 + \alpha\alpha^\intercal)\det(\lambda I - A_2 + \beta\beta^\intercal) \\ &\quad - \alpha^\intercal \operatorname{adj}(\lambda I - A_1 + \alpha\alpha^\intercal)\alpha \det(\lambda I - A_2 + \beta\beta^\intercal) \\ &\quad - \det(\lambda I - A_1 + \alpha\alpha^\intercal)\beta^\intercal \operatorname{adj}(\lambda I - A_2 + \beta\beta^\intercal)\beta \\ &= \det(\lambda I - A_1 + \alpha\alpha^\intercal)\det(\lambda I - A_2 + \beta\beta^\intercal) \\ &\quad - (\det(\lambda I - A_1 + \alpha\alpha^\intercal) - \det(\lambda I - A_1))\det(\lambda I - A_2 + \beta\beta^\intercal) \\ &\quad - \det(\lambda I - A_1 + \alpha\alpha^\intercal)(\det(\lambda I - A_2 + \beta\beta^\intercal) - \det(\lambda I - A_2)) \\ &= \det(\lambda I - A_1)\det(\lambda I - A_2 + \beta\beta^\intercal) + \det(\lambda I - A_1 + \alpha\alpha^\intercal)\det(\lambda I - A_2) \\ &\quad - \det(\lambda I - A_1 + \alpha\alpha^\intercal)\det(\lambda I - A_2 + \beta\beta^\intercal). \end{aligned}$$

Note that there are infinite choices for $\lambda$ such that $B$ is invertible and both sides the last equality are univariate polynomials in $\lambda$. Thus, the formula (2) holds.

Now, set $\lambda = \tau + \epsilon$, with $\epsilon > 0$. Since $\min\{\lambda_{\min}(\alpha\alpha^\intercal - I - A_1), \lambda_{\min}(\beta\beta^\intercal - I - A_2)\} \geqslant -1-\tau$, it follows that $B$ is positive definite. Therefore, there exists an invertible matrix $P_1$ such that $P_1 B P_1^\intercal = I$. Let $C = P_1\mathbf{r}\mathbf{r}^\intercal P_1^\intercal$. Since the rank of $C$ is equal to the rank of $\mathbf{r}\mathbf{r}^\intercal$, it is not larger than one. This implies the existence of another orthogonal matrix $P_2$ such that $P_2 C P_2^\intercal = \operatorname{diag}(\mu, 0, \ldots, 0)$. Let $P = P_2 P_1$. Then

$$P(\lambda I - A)P^\intercal = PBP^\intercal - P\mathbf{r}\mathbf{r}^\intercal P^\intercal = P_2(P_1 B P_1^\intercal)P_2^\intercal - P_2 C P_2^\intercal = I - \operatorname{diag}(\mu, 0, \ldots, 0).$$

Since $\lambda I - A$ and $I - \operatorname{diag}(\mu, 0, \ldots, 0)$ are congruent, they have the same inertia indices. Hence, $\lambda_{n-1}(\lambda I - A) = \lambda - \lambda_2(A) = \tau + \epsilon - \lambda_2(A) > 0$, where $n$ is the order of $A$. From $\epsilon > 0$, we have $\lambda_2(A) \leqslant \tau$.

It remains to consider the equality case. For $\lambda_{\min}(\alpha\alpha^\intercal - I - A_1) = \lambda_{\min}(\beta\beta^\intercal - I - A_2) = -1-\tau$, it holds $\det(\tau I - A_1 + \alpha\alpha^\intercal) = \det(\tau I - A_2 + \beta\beta^\intercal) = 0$. From (2), we have $\det(\tau I - A) = 0$ and $\lambda_2(A) = \tau$, since by the statement assumption there is $\lambda_1(A) > \tau$. □

We prove the following consequence.

**Corollary 6 (Sufficiency in Theorem 2).** *Every graph of* $\mathcal{D} \cup \{\overline{D}_{2,4} \vee K_{52}\}$ *is a* $\sigma$*-graph.*

*Proof.* First, the inequality $\lambda_2(\overline{D}_{2,4} \vee K_{52}) \leqslant \sigma$ is confirmed by direct computation. Let $G \in \mathcal{D}$, i.e., $G \cong H_s \vee (K_s | W_{s,s})$, for $s \geqslant 1$.

Suppose that the order of $H_s$ is $k$. By taking $\alpha = \mathbf{j}_k$ and $\beta^\intercal = (\mathbf{j}_s^\intercal, \mathbf{0}_s^\intercal)$, we arrive at

$$A(G) = \begin{pmatrix} A(H_s) & \alpha\beta^\intercal \\ \beta\alpha^\intercal & A(W_{s,s}) \end{pmatrix} = \begin{pmatrix} A(H_s) & J_{k\times s} & O \\ J_{s\times k} & J_s - I_s & I_s \\ O & I_s & O \end{pmatrix}.$$

Since

$$\det\Big(\lambda I - \big(\beta\beta^\intercal - I - A(W_{s,s})\big)\Big) = \det\begin{pmatrix} \lambda I_s & -I_s \\ -I_s & (\lambda+1)I_s \end{pmatrix} = (\lambda^2+\lambda-1)^s,$$

by employing Lemma 4(i), we obtain $\lambda_{\min}(\beta\beta^\intercal - I - A(W_{s,s})) = -1-\sigma$.

Note that $\Phi(K_1 \vee C_5) = (\lambda^2+\lambda-1)^2(\lambda^2-2\lambda-5)$, $\Phi(\overline{W}_{3,3}) = (\lambda^2+\lambda-1)^2(\lambda^2-2\lambda-4)$, $\Phi(D_{2,3}) = (\lambda+1)^3(\lambda^3-3\lambda^2-3\lambda+7)$ and

$$\begin{aligned} \Phi(W_{s,s}) &= \det\begin{pmatrix} (\lambda+1)I_s - J_s & -I_s \\ -I_s & \lambda I_s \end{pmatrix} \\ &= \det(\lambda I_s)\det\big[(\lambda+1)I_s - J_s - \lambda^{-1}I_s\big] = (\lambda^2+\lambda-1)^{s-1}\,(\lambda^2-(s-1)\lambda-1), \end{aligned}$$

where the last equality holds since the eigenvalues of $J_s$ are $s$ and $s-1$ copies of 0. Since the least root of these four polynomials is not less than $-1-\sigma$ and $\overline{H}_s = s\big((K_1 \vee C_5) \cup \overline{W}_{3,3} \cup D_{2,3} \cup W_{s,s}\big)$, we deduce $\lambda_{\min}(A(\overline{H}_s)) \geqslant -1-\sigma$. Combining $\lambda_{\min}(\beta\beta^\intercal - I - A(W_{s,s})) = -1-\sigma$ with

$$\lambda_{\min}(\alpha\alpha^\intercal - I - A(H_s)) = \lambda_{\min}(A(\overline{H}_s)) \geqslant -1-\sigma,$$

we obtain $\lambda_2(G) \leqslant \sigma$ by Theorem 5. □

## 4 Necessity

Let $R_1, R_2, \ldots, R_7$ be the graphs illustrated in Figure 1. One can easily check that none of them is a $\sigma$-graph. In other words, these graphs do not appear as induced subgraphs of $\sigma$-graphs, or they are forbidden induced subgraphs for $\sigma$-graphs.

One more term is needed. If $D_{a,b}$, with either $3 = a \leqslant b \leqslant 5$ or $a = 2 < 4 \leqslant b \leqslant 18$, is a component of $\overline{G}$, then we say that $D_{a,b}$ is a *special component* of $\overline{G}$.

The following theorem features as another meaningful result of this paper. It plays a crucial role in the proof of Theorem 2.

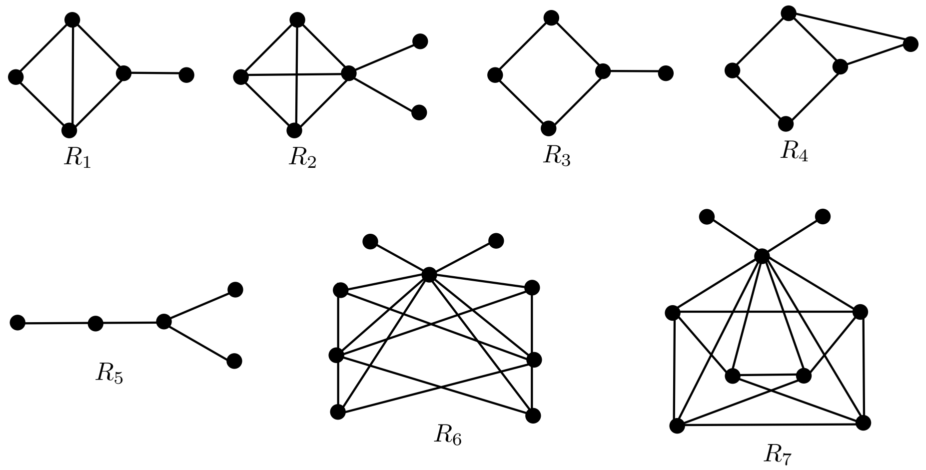


Figure 1: Forbidden induced subgraphs for $\sigma$-graphs.

**Theorem 7.** *Let $G$ be a connected graph. If $\overline{G}$ contains no special component and $\omega(G) \geqslant 8$, then $\lambda_2(G) \leqslant \sigma$ if and only if either*

*(i) $G$ is an induced subgraph of $\mathcal{D}$, or*

*(ii) $\overline{G}$ is an induced subgraph of some graph of $\{D_{2,27} \cup 6K_1, D_{2,21} \cup W_{6,6}, D_{2,21} \cup \overline{W}_{3,3} \cup 3K_1, D_{2,21} \cup C_5 \cup 4K_1, D_{2,19} \cup K_4 \cup 5K_1\}$, or*

*(iii) $\overline{G} \subset D_{4,4} \cup H$, for $H \in \{\overline{W}_{3,3} \cup 3K_1, C_5 \cup 4K_1, W_{6,6}, K_4 \cup 5K_1\}$, or*

*(iv) $\overline{G} \subset D_{3,6} \cup H$, for $H \in \{\overline{W}_{3,3} \cup 3K_1, C_5 \cup 4K_1, W_{6,6}\}$.*

The proof is based on a sequence of facts and lemmas. In what follows, $G$ denotes a graph of the previous theorem. We always suppose that $\omega(G) = q$ and $K_q$ is a fixed maximum clique of $G$, with $V(K_q) = \{w_1, w_2, \dots, w_q\}$. We also denote by $V_1$ (resp. $V_2$) the set of vertices of $V(G) \setminus V(K_q)$, such that each of them is adjacent to exactly one (more than one) vertex of $K_q$. In the first four facts, we do not need the assumption $q \geqslant 8$ (as formulated in the previous theorem); this assumption will be imposed after Lemma 13.

The first fact tells us that $V(G) = V(K_q) \cup V_1 \cup V_2$, for $q \geqslant 3$.

**Fact 8.** *If $q \geqslant 3$, then for every $x \in V(G) \setminus V(K_q)$, $x$ is adjacent to at least one vertex of $K_q$.*

*Proof.* For a contradiction, assume that $G$ contains some vertex, which is not a neighbour of any vertex of $K_q$. Since $G$ is connected, we can choose a vertex $x$ of $G$ such that $x$ is not adjacent to any vertex of $K_q$, $xy \in E(G)$ for $y \in V(G) \setminus V(K_q)$, and $y$ is adjacent to some vertices of $K_q$. Since $2K_2 \subset G$ when $|N_{K_q}(y)| = 1$ and $R_1 \subset G$ when $|N_{K_q}(y)| \geqslant 2$, we have $\lambda_2(G) > \sigma$, which is a contradiction. $\square$

We proceed with the following, more or less simple, facts. They lead to the forthcoming Lemma 13.

**Fact 9.** *Let* $k = |N_{K_q}(x)|$, *for* $x \in V_2$. *We have* $1 \leqslant q - k \leqslant 3$, *where* $q = k + 3$ *implies* $q \in \{5, 6\}$. *In other words, if* $q \geqslant 7$, *then* $q - 2 \leqslant k \leqslant q - 1$.

*Proof.* Since $\omega(G) = q$, it follows that $k \leqslant q - 1$. Let $H = K_1 \vee (K_k|K_q)$. Since $H \subset G$ and $x \in V_2$, we have $\lambda_2(H) \geqslant \lambda_2(K_1 \vee (K_2|K_6)) > \sigma$, whenever $2 \leqslant k \leqslant q - 4$. Therefore, $q \leqslant k + 3$. If $q = k + 3 \geqslant 7$, again $\lambda_2(H) \geqslant \lambda_2(K_1 \vee (K_4|K_7)) > \sigma$ leads to the impossible scenario. Hence, $q \in \{5, 6\}$, as $k = 1$ is eliminated by definition of $V_2$. □

**Fact 10.** *Let* $x_1, x_2 \in V(G) \setminus V(K_q)$ *where* $x_i$ *is adjacent to exactly* $k_i$ *vertices of* $K_q$, *for* $1 \leqslant i \leqslant 2$. *If* $q \geqslant 2 + k_1 + k_2$, *then* $x_1$ *is not adjacent to* $x_2$.

*Proof.* If $x_1x_2 \in E(G)$, then $2K_2 \subset G$ (as $q \geqslant 2 + k_1 + k_2$), which is not in line with $\lambda_2(G) \leqslant \sigma$. □

We recall the reader that the independent set is a vertex subset that do not contain adjacent vertices. In other words, it is the complement of a clique.

**Fact 11.** *If* $q \geqslant 4$, *then each vertex of* $K_q$ *is adjacent to at most one vertex of* $V_1$, *and* $V_1$ *is an independent set of* $G$.

*Proof.* Assume for a contradiction that there exists a vertex $x \in V(K_q)$ adjacent to at least two vertices of $V_1$. In this case, either $2K_2 \subset G$ or $R_2 \subset G$, which contradicts $\lambda_2(G) \leqslant \sigma$. Combining this with Fact 10, we arrive at the desired result. □

**Fact 12.** *If* $q \geqslant 8$ *and* $y \in V_1$ *with* $N_{K_q}(y) = \{z\}$, *then* $z$ *is adjacent to every vertex of* $V_2$ *and* $y$ *is a pendant vertex of* $G$.

*Proof.* Suppose that $x \in V_2$. By Fact 9, we have $q - 2 \leqslant |N_{K_q}(x)| \leqslant q - 1$.

If $z \notin N_{K_q}(x)$, then $xy \in E(G)$, as $R_1 \not\subset G$. Thus, we have $\overline{K_2 \cup K_{1,7}} \subset G$ for $|N_{K_q}(x)| = q - 1 \geqslant 7$ or $R_4 \subset G$ for $|N_{K_q}(x)| = q - 2$, contradicting $\lambda_2(G) \leqslant \sigma$. Therefore, $z \in N_{K_q}(x)$, which implies that $z$ is adjacent to every vertex of $V_2$.

Since $R_1 \not\subset G$ and $q - |N_{K_q}(x)| \geqslant 1$ with $q \geqslant 8$, we have $xy \notin E(G)$. Now, Facts 8 and 11 imply that $y$ is a pendant vertex of $G$. □

At this point we single out one possibility for $V_2$.

**Lemma 13.** *If* $|V_2| \leqslant 1$ *and* $q \geqslant 8$, *then* $G$ *is an induced subgraph of* $W_{q,q}$ *or* $(K_1 \cup K_2) \vee (K_{q-1}|W_{q-1,q-1})$. *Consequently,* $G$ *is an induced subgraph of some graph of* $\mathcal{D}$.

*Proof.* If $|V_2| = 0$, then $G \subset W_{q,q}$ by Fact 11. Let $V_2 = \{x\}$. By Fact 9, we have $q - 2 \leqslant |N_{K_q}(x)| \leqslant q - 1$. By Fact 12, $G$ is an induced subgraph of $(K_1 \cup K_2) \vee (K_{q-1}|W_{q-1,q-1}) \subset (\overline{K_1 \vee C_5}) \vee (K_{q-1}|W_{q-1,q-1})$.

In any case, $G$ is an induced subgraph of some graph of $\mathcal{D}$, as follows by definition of the latter class. □

Henceforth, we always suppose that $V_2$ contains at least two vertices and that $q \geqslant 8$ holds. Further, by Fact 9, we have $q-2 \leqslant |N_{K_q}(x)| \leqslant q-1$, for every $x \in V_2$. Accordingly, we set

$$S_j = \{x : x \in V_2, |N_{K_q}(x)| = j\}, \text{ for } j \in \{q-2, q-1\}.$$

For fixed vertices $w_1, w_2$ of $V(K_q)$, we define

$$X_{w_1,w_2} = \{x : x \in S_{q-2}, w_1 \notin N_{K_q}(x) \text{ and } w_2 \notin N_{K_q}(x)\}$$

and

$$Y_w = \{x : x \in S_{q-1}, w \notin N_{K_q}(x)\}.$$

Observe that $Y_w$ is an independent set of $G$, for every $w \in V(K_q)$, as $q = \omega(G)$.

We establish some structural characterizations of $G$. Facts 14–18 deal with $X_{w,w'}$ and $Y_w$.

**Fact 14.** *If $w_1$, $w_2$ are distinct vertices of $V(K_q)$ with $z_1 \in Y_{w_1}$ and $z_2 \in Y_{w_2}$, then $N_{X_{w_1,w_2}}(z_1) = N_{X_{w_1,w_2}}(z_2)$. Moreover, if $z_1z_2 \in E(G)$, then $N_{X_{w_1,w_2}}(z_1) = N_{X_{w_1,w_2}}(z_2) = \emptyset$.*

*Proof.* By the symmetry, we may suppose that $u \in X_{w_1,w_2}$ with $z_1u \in E(G)$ and $z_2u \notin E(G)$. Then either $G[u, z_1, z_2, w_1] \cong 2K_2 \subset G$ for $z_1z_2 \notin E(G)$ or $G[u, z_1, z_2, w_1, w_2] \cong R_3 \subset G$ for $z_1z_2 \in E(G)$. Thus, $N_{X_{w_1,w_2}}(z_1) = N_{X_{w_1,w_2}}(z_2)$.

Next, if $z_1z_2 \in E(G)$ and $u \in N_{X_{w_1,w_2}}(z_1) = N_{X_{w_1,w_2}}(z_2) \neq \emptyset$, then $G[u, z_1, z_2, w_1, w_2] \cong R_4$. Hence, $N_{X_{w_1,w_2}}(z_1) = N_{X_{w_1,w_2}}(z_2) = \emptyset$. □

**Fact 15.** *If $w_1, w_2$ are distinct vertices of $V(K_q)$ with $\max\{|Y_{w_1}|, |Y_{w_2}|\} \geqslant 2$, then every vertex of $Y_{w_1}$ is adjacent to every vertex of $Y_{w_2}$, and there is no edge between $T$ and $X_{w_1,w_2}$, where $T = Y_{w_1} \cup Y_{w_2}$.*

*Proof.* We may suppose that $|Y_{w_1}| \geqslant |Y_{w_2}| \geqslant 0$ with $z_1, z_2 \in Y_{w_1}$. Recall that $Y_{w_1}$ and $Y_{w_2}$ are independent sets of $G$. If $z_3 \in Y_{w_2}$ and either $z_1z_3 \notin E(G)$ or $z_2z_3 \notin E(G)$, then $G[w_1, w_2, z_1, z_2, z_3] \in \{R_3, R_5\}$. Hence, every vertex of $Y_{w_1}$ is adjacent to every vertex of $Y_{w_2}$.

Suppose that $u \in X_{w_1,w_2}$. If $\{z_1u, z_2u\} \cap E(G) \neq \emptyset$, then $G[w_1, w_2, z_1, z_2, u] \in \{R_3, R_5\}$. If $z_3 \in Y_{w_2} \neq \emptyset$ and $z_3u \in E(G)$, then $G[w_1, w_2, z_1, z_3, u] \cong R_3$. Hence, there is no edge between $T$ and $X_{w_1,w_2}$. □

**Fact 16.** *For any pair of vertices $w_1$, $w_2 \in V(K_q)$, $X_{w_1,w_2}$ is an independent set of $G$ and $|X_{w_1,w_2}| \leqslant 2$. Moreover, if $|X_{w_1,w_2}| = 2$ and $z \in Y_{w_1} \cup Y_{w_2}$, then $|N(z) \cap X_{w_1,w_2}| = 1$.*

*Proof.* Since $2K_2 \not\subset G$, $X_{w_1,w_2}$ is an independent set of $G$. Since $q \geqslant 8$ and $K_3 \vee (3K_1 \cup K_2) \not\subset G$, we have $|X_{w_1,w_2}| \leqslant 2$. Next, suppose that $|X_{w_1,w_2}| = 2$. Since $K_3 \vee (2K_1 \cup W_{2,1}) \not\subset G$, we also have $|N(z) \cap X_{w_1,w_2}| \geqslant 1$. On the other hand, we have $|N(z) \cap X_{w_1,w_2}| \leqslant 1$ as $R_5 \not\subset G$. Thus, $|N(z) \cap X_{w_1,w_2}| = 1$. □

From Facts 15 and 16, we immediately obtain the following one.

**Fact 17.** *If $w_1, w_2$ are distinct vertices of $V(K_q)$ with* $\max\{|Y_{w_1}|, |Y_{w_2}|\} \geqslant 2$*, then we have* $|X_{w_1,w_2}| \leqslant 1$*.*

**Fact 18.** *Let $w_1, w_2, w_3$ be distinct vertices of $V(K_q)$ and $z_j \in Y_{w_j}$ for $j \in \{1, 2, 3\}$. If $z_1z_2 \in E(G)$, then $|N(z_3) \cap \{z_1, z_2\}| \geqslant 1$.*

*Proof.* If $|N(z_3) \cap \{z_1, z_2\}| = 0$, then $G[z_1, z_2, z_3, w_2, w_3] \cong R_1$. Therefore, we have $|N(z_3) \cap \{z_1, z_2\}| \geqslant 1$. □

Now, we include sets $S_j$ (defined above).

**Fact 19.** *If $x, y \in S_{q-2}$, then $q - 4 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q - 2$, along with $xy \notin E(G)$ for $q - 3 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q - 2$ and $xy \in E(G)$ for $|N_{K_q}(x) \cap N_{K_q}(y)| = q - 4$.*

*Proof.* Under the given assumptions, we have $q - 4 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q - 2$.

If $|N_{K_q}(x) \cap N_{K_q}(y)| = q-2$, then $xy \notin E(G)$, as $2K_2 \not\subset G$. If $|N_{K_q}(x) \cap N_{K_q}(y)| = q-3$, then $xy \notin E(G)$, as $R_4 \not\subset G$. If $|N_{K_q}(x) \cap N_{K_q}(y)| = q-4$, then $xy \in E(G)$, as $R_1 \not\subset G$. □

**Fact 20.** *If $x \in S_{q-2}$ and $y \in S_{q-1}$, then $q - 3 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q - 2$, where $|N_{K_q}(x) \cap N_{K_q}(y)| = q - 3$ implies $xy \in E(G)$.*

*Proof.* Since $x \in S_{q-2}$ and $y \in S_{q-1}$, we have $q - 3 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q - 2$. If $|N_{K_q}(x) \cap N_{K_q}(y)| = q - 3$, then $xy \in E(G)$, as $R_1 \not\subset G$. □

**Fact 21.** *If $x, y \in S_{q-1}$, then $q-2 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q-1$, where $|N_{K_q}(x) \cap N_{K_q}(y)| = q - 1$ implies $xy \notin E(G)$.*

*Proof.* We have $q - 2 \leqslant |N_{K_q}(x) \cap N_{K_q}(y)| \leqslant q - 1$. If $|N_{K_q}(x) \cap N_{K_q}(y)| = q - 1$, then $xy \notin E(G)$, as $q = \omega(G)$. □

In Facts 22–24, we fix some vertices of $K_q$ and consider the interplay between $X_{w_1,w_2}$s and $Y_{w_1}$s.

**Fact 22.** *Let $w_1, w_2, w_3$ be distinct vertices of $K_q$. Then:*

*(i)* $|X_{w_1,w_2} \cup X_{w_2,w_3}| \leqslant 2$*;*

*(ii) If* $\min\{|Y_{w_1}|, |X_{w_2,w_3}|\} \geqslant 1$*, then $X_{w_1,w_2} = \emptyset$ and every vertex of $Y_{w_1}$ is adjacent to every vertex of $Y_{w_2}$;*

*(iii) if $x \in X_{w_1,w_2}$, $y \in X_{w_2,w_3}$ and $z \in Y_{w_2}$, then $|N(z) \cap \{x, y\}| \in \{0, 2\}$.*

*Proof.* (i): By Facts 16 and 19, we have that $X_{w_1,w_2} \cup X_{w_2,w_3}$ is an independent set of $G$, along with $\max\{|X_{w_1,w_2}|, |X_{w_2,w_3}|\} \leqslant 2$. If $z_1, z_2, z_3$ are distinct vertices of $X_{w_1,w_2} \cup X_{w_2,w_3}$, then $G[z_1, z_2, z_3, w_1, w_3]$ is $R_5$. Hence, $|X_{w_1,w_2} \cup X_{w_2,w_3}| \leqslant 2$.

(ii): Let $x \in Y_{w_1}$ and $y \in X_{w_2,w_3}$. Assume that $z \in X_{w_1,w_2}$. By Facts 19 and 20, we have $xy \in E(G)$ and $yz \notin E(G)$. This implies $G[x, y, z, w_1, w_3] \cong R_3$ for $xz \notin E(G)$ and $G[x, y, z, w_1, w_3] \cong R_4$ for $xz \in E(G)$. Hence, $X_{w_1,w_2} = \emptyset$. Next, suppose that $z \in Y_{w_2}$. If $xz \notin E(G)$, we obtain $G[x, y, z, w_1, w_2] \cong R_3$ for $yz \notin E(G)$ and $G[x, y, z, w_1, w_2] \cong R_4$ for $yz \in E(G)$. Hence, $xz \in E(G)$.

(iii): From $x \in X_{w_1,w_2}$ and $y \in X_{w_2,w_3}$, we have $xy \notin E(G)$ by Fact 19. If $|N(z) \cap \{x, y\}| = 1$, then $G[x, y, z, w_1, w_3] \cong R_1$. Hence, $|N(z) \cap \{x, y\}| \in \{0, 2\}$. □

**Fact 23.** $\min\{|X_{w_1,w_2}|, |X_{w_3,w_4}|, |X_{w_1,w_3}|\} = 0$ *holds for distinct vertices* $w_1, w_2, w_3, w_4$ *of* $K_q$.

*Proof.* Assume that $x_1 \in Y_{w_1,w_2}$, $y_1 \in X_{w_3,w_4}$ and $y_2 \in X_{w_1,w_3}$. By Fact 19, we have $x_1y_1 \in E(G)$ and $y_2 \notin N_G(x_1) \cup N_G(y_1)$. This implies $G[w_2, w_3, x_1, y_1, y_2] \cong R_3$, and we are done. □

**Fact 24.** *If* $\min\{|Y_{w_1}|, |Y_{w_2}|\} \geqslant 1$ *with* $|Y_{w_1}| + |Y_{w_2}| \geqslant 4$ *and* $|X_{w_1,w_2}| \geqslant 1$*, then* $V_1 = \emptyset$.

*Proof.* By Fact 15, every vertex of $Y_{w_1}$ is adjacent to every vertex of $Y_{w_2}$, and every vertex of $X_{w_1,w_2}$ is not adjacent to any vertex of $Y_{w_1} \cup Y_{w_2}$. Assume that $z \in V_1$ with $N_{K_q}(z) = \{w_3\}$. By employing Fact 12, we deduce that $w_3 \notin \{w_1, w_2\}$ and $R_6$ or $R_7$ is an induced subgraph of $G[U]$ with $U = \{w_1, w_2, w_3, z\} \cup Y_{w_1} \cup Y_{w_2} \cup X_{w_1,w_2}$. This contradiction leads to $V_1 = \emptyset$. □

We recall from Section 2 that $N_Y[x]$ denotes the closed neighbourhood of $x$ in $Y$. Let

$$N_Y(X) = \{y : y \in Y \text{ and } |N_X(y)| > 0\} \quad \text{and} \quad N_Y[X] = N_Y(X) \cup X.$$

We also set $\widetilde{N}_Y(X) = \{y : y \in Y \text{ and } N_X(y) \neq X\}$.

By Facts 11 and 12, every vertex of $V(G) \setminus N_{K_q}[V_1]$ is adjacent to every vertex of $N_{K_q}(V_1)$, which implies that $G \cong G_0 \vee (K_h|W_{h,h})$, where $G_0 = G \setminus N_{K_q}[V_1]$ and $|V_1| = h$. In the remainder of this section, $G_0$ is always as in the previous sentence. The next fact gives the structure of components of $\overline{G}_0$.

**Fact 25.** *If* $H$ *is a component of* $\overline{G}_0$*, then either* $H \cong W_{t,j}$ *with* $0 \leqslant j \leqslant t$ *and* $t \geqslant 1$*, or* $H \cong D_{a,b}$ *with* $b \geqslant a \geqslant 2$*, or* $H \in \{K_1 \vee P_3, K_1 \vee P_4, K_1 \vee C_5, C_5, \overline{W}_{3,3}\}$*. Moreover, if* $H \cong D_{a,b}$ *with* $a + b \geqslant 6$*, then* $V_1 = \emptyset$ *with* $G \cong G_0$.

*Proof.* To avoid ambiguity, when we say that two vertices are adjacent, we always mean they are adjacent in $G_0$. When $V(H) \cap V_2 = \emptyset$, we have $V(H) \subseteq V(K_q)$. Thus $H \cong K_1 \cong W_{1,0}$. Next, we may suppose that $V(H) \cap V_2 \neq \emptyset$. Since $V_2 = S_{q-1} \cup S_{q-2}$ by Fact 9, we need to consider the following three cases:

*C*ase 1: $V(H) \cap S_{q-2} = \emptyset$.

This implies that $V(H) \cap S_{q-1} \neq \emptyset$. Let $x_1 \in V(H) \cap S_{q-1}$. Then there exists a unique vertex $w_{k_1} \in V(K_q)$, such that $x_1w_{k_1} \in E(H)$. Since $Y_{w_{k_1}}$ is an independent set of $G$, we have $Y_{w_{k_1}} \cup \{w_{k_1}\} \subseteq V(H)$.

If $V(H) = Y_{w_{k_1}} \cup \{w_{k_1}\}$, then $H \cong K_t \cong W_{t,0}$, where $t = |Y_{w_{k_1}}| + 1 \geqslant 2$. Next, we suppose that $V(H) \neq Y_{w_{k_1}} \cup \{w_{k_1}\}$.

If $|Y_{w_{k_1}}| \geqslant 2$, then every vertex of the independent set $Y_{w_{k_1}}$ is adjacent to every vertex of $S_{q-1} \setminus Y_{w_{k_1}}$ by Fact 15. This implies $V(H) = Y_{w_{k_1}} \cup \{w_{k_1}\}$, which is a contradiction. Hence, $Y_{w_{k_1}} = \{x_1\}$.

Since $V(H) \neq Y_{w_{k_1}} \cup \{w_{k_1}\}$, $x_1$ is not adjacent to some vertex $y_1 \in Y_{w_h}$, where $h \neq k_1$. By Fact 15, we have $|Y_{w_h}| = 1$. Let $S$ be a maximum independent set of $G_0[S_{q-1}]$ containing $x_1$. Then, $|S| \geqslant 2$. By the maximality of $|S|$ and Fact 18, we conclude that every vertex of $S' = S_{q-1} \setminus S$ (if not empty) is adjacent to every vertex of $S$. Since $|Y_{w_{k_1}}| = 1$ holds for every $w_1 \in \widetilde{N}_{K_q}(S)$ by Fact 15, we have $Y_{w_{k_1}} \subset S$. Therefore, every vertex of $S' = S_{q-1} \setminus S$ is adjacent to every vertex of $\widetilde{N}_{K_q}(S)$. This implies $V(H) = S \cup \widetilde{N}_{K_q}(S)$, and so $H \cong W_{t,t}$, where $t = |S| \geqslant 2$.

*C*ase 2: $V(H) \cap S_{q-1} = \emptyset$.

Let $x_1 \in V(H) \cap S_{q-2}$. Then there exists a unique pair $\{w_{k_1}, w_{l_1}\} \subset V(K_q)$ such that $x_1 \in X_{w_{k_1},w_{l_1}}$, that is, $x_1w_{k_1}, x_1w_{l_1} \in E(H)$. If $V(H) = X_{w_{k_1},w_{l_1}} \cup \{w_{k_1}, w_{l_1}\}$, then $H$ is isomorphic to $W_{2,1}$ or $K_1 \vee P_3$, as $X_{w_{k_1},w_{l_1}}$ is an independent set and $|X_{w_{k_1},w_{l_1}}| \leqslant 2$ by Fact 16. Next, suppose that $V(H) \neq X_{w_{k_1},w_{l_1}} \cup \{w_{k_1}, w_{l_1}\}$. Since $V(H) \cap S_{q-1} = \emptyset$, there must exist $y_1 \in X_{w_{k_2},w_{l_2}}$ with $\{k_2, l_2\} \neq \{k_1, l_1\}$ such that $y_1$ is not adjacent to some vertex of $\{w_{k_1}, w_{l_1}\} \cup X_{w_{k_1},w_{l_1}}$.

By Fact 19, we have $|\{k_2, l_2\} \cap \{k_1, l_1\}| = 1$, and thus we may suppose that $k_1 = k_2$ and $l_2 \neq l_1$. Now, since $|\{k_2, l_2\} \cap \{k_1, l_1\}| = 1$, we have $X_{w_{k_1},w_{l_1}} = \{x_1\}$, $X_{w_{k_1},w_{l_2}} = \{y_1\}$ and $x_1y_1 \notin E(G)$ by Facts 19 and 22(i). If $V(H) = \{x_1, y_1, w_{k_1}, w_{l_1}, w_{l_2}\}$, then $H \cong W_{3,2}$.

Next, suppose that $V(H) \neq \{x_1, y_1, w_{k_1}, w_{l_1}, w_{l_2}\}$. Then there must exist $z_1 \in X_{w_{k_3},w_{l_3}}$ such that $z_1 \notin \{x_1, y_1\}$ and $z_1$ is not adjacent to some vertex of $\{x_1, y_1, w_{k_1}, w_{l_1}, w_{l_2}\}$. We claim that

$$|\{k_1, l_1\} \cap \{k_3, l_3\}| = 1 = |\{k_1, l_2\} \cap \{k_3, l_3\}|. \tag{3}$$

Indeed, by assuming that $|\{k_1, l_1\} \cap \{k_3, l_3\}| = 0$, on the basis of $\min\{|X_{w_{k_1},w_{l_1}}|, |X_{w_{k_3},w_{l_3}}|, |X_{w_{k_1},w_{l_2}}|\} \geqslant 1$ and Fact 23, we deduce $l_2 \notin \{k_3, l_3\}$, and thus $|\{k_1, l_2\} \cap \{k_3, l_3\}| = 0$. Now, from Fact 19, $z_1$ is adjacent to every vertex belonging to $\{x_1, y_1, w_{k_1}, w_{l_1}, w_{l_2}\}$, which is impossible, and this confirms (3).

We first suppose that $k_1 \notin \{k_3, l_3\}$. By (3), we have $\{k_3, l_3\} = \{l_1, l_2\}$ and thus $X_{w_{l_1},w_{l_2}} = \{z_1\}$ by Fact 22(i). We claim that $V(H) = \{x_1, y_1, z_1, w_{k_1}, w_{l_1}, w_{l_2}\}$. Otherwise, there exists $z_2 \in X_{w_{k_4},w_{l_4}}$ such that $z_2 \notin \{x_1, y_1, z_1\}$ and $z_2$ is not adjacent to some vertex of $\{x_1, y_1, z_1, w_{k_1}, w_{l_1}, w_{l_2}\}$. If $z_2$ is adjacent to every vertex of $\{x_1, y_1, w_{k_1}, w_{l_1}, w_{l_2}\}$, then $\{k_4, l_4\} \cap \{k_1, l_1, l_2\} = \emptyset$ by Fact 19, and thus $z_1z_2 \in E(G)$ holds by the same fact, which contradicts the choice of $z_2$. Thus, $z_2$ is not adjacent to some vertex of $\{x_1, y_1, w_{k_1}, w_{l_1}, w_{l_2}\}$. By the symmetry of $z_1$ and $z_2$, we have $k_1 \in \{k_4, l_4\}$ by (3) (replacing $\{k_3, l_3\}$ with $\{k_4, l_4\}$) and $\{k_4, l_4\} \cap \{l_1, l_2\} = \emptyset$, as $X_{w_{l_1},w_{l_2}} = \{z_1\}$ and $z_2 \notin \{x_1, y_1, z_1\}$. Suppose that $k_4 = k_1$. Then $\min\{|X_{w_{k_1},w_{l_4}}|, |X_{w_{l_1},w_{l_2}}|, |X_{w_{k_1},w_{l_1}}|\} \geqslant 1$, contradicting Fact 23. This confirms our claim that $V(H) = \{x_1, y_1, z_1, w_{k_1}, w_{l_1}, w_{l_2}\}$. By Fact 19, $\{x_1, y_1, z_1\}$ is an independent set of $G$ and thus $H \cong \overline{W}_{3,3}$.

Secondly, we suppose that $k_1 \in \{k_3, l_3\}$. By (3), we have $k_3 = k_1$ and $l_3 \notin \{l_1, l_2\}$. By Facts 19 and 22 (i), we have $X_{w_{k_1},w_{l_3}} = \{z_1\}$ with $x_1, y_1 \notin N_G(z_1)$. Since $\{x_1, y_1, z_1, w_{k_1}, w_{l_1}, w_{l_2}, w_{l_3}\} \subseteq V(H)$, there exists $c \geqslant 3$, such that $c$ is as large as possible and $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\} \subseteq V(H)$, where $X_{w_{k_1},w_{s_i}} = \{x_{s_i}\}$ for $1 \leqslant i \leqslant c$.

Next, we show that

$$V(H) = \{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\}. \tag{4}$$

Otherwise, there exists $z_2 \in X_{w_f,w_g}$ such that $z_2 \notin \{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\}$ and $z_2$ is not adjacent to some vertex of $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\}$. By Fact 19, we have $|\{f, g\} \cap$

$\{k_1, s_1, s_2, \ldots, s_c\}| \geqslant 1$. By the choice of $c$ and $z_2 \notin \{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\}$, we have $k_1 \notin \{f, g\}$. Suppose that $f = s_1$ and $g \neq s_2$, as $c \geqslant 3$. Thus $\min\{|X_{w_{k_1},w_{s_2}}|, |X_{w_{s_1},w_g}|, |X_{w_{k_1},w_{s_1}}|\} \geqslant 1$, which contradicts Fact 23. Hence, (4) holds.

By Fact 19, $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\}$ is an independent set of $G$, and thus $H \cong W_{t,t-1}$ with $t = c + 1 \geqslant 4$.

*C*ase 3: $V(H) \cap S_{q-2} \neq \emptyset$ and $V(H) \cap S_{q-1} \neq \emptyset$.

We show that $H$ contains a pair of vertices $x_1, y_1$, such that $x_1 \in Y_{w_{k_1}}$ and $y_1 \in X_{w_{l_1},w_{r_1}}$ with $k_1 \in \{l_1, r_1\}$. Suppose that $k_1 \notin \{l_1, r_1\}$ holds for every pair of vertices $x_1 \in Y_{w_{k_1}}$ and $y_1 \in X_{w_{l_1},w_{r_1}}$. Take $V_{2,1} = S_{q-1} \cup \widetilde{N}_{K_q}(S_{q-1})$ and $V_{2,2} = S_{q-2} \cup \widetilde{N}_{K_q}(S_{q-2})$. By Fact 20, every vertex of $V_{2,1}$ is adjacent to every vertex of $V_{2,2}$. Thus, $\overline{G_0[V_{2,1} \cup V_{2,2}]}$ is disconnected, but this contradicts $V(H) \cap S_{q-2} \neq \emptyset$ and $V(H) \cap S_{q-1} \neq \emptyset$. By symmetry, we may suppose that $k_1 = r_1$, that is, $x_1 \in Y_{w_{k_1}}$ and $y_1 \in X_{w_{k_1},w_{l_1}}$. Let $U = \{w_{k_1}, w_{l_1}\} \cup Y_{w_{k_1}} \cup X_{w_{k_1},w_{l_1}}$.

We first suppose that $U = V(H)$. If $Y_{w_{k_1}} \cup X_{w_{k_1},w_{l_1}}$ is an independent set of $G$, then $|X_{w_{k_1},w_{l_1}}| = 1$ by Fact 16, and thus $H \cong W_{t,1}$ with $t = |Y_{w_{k_1}}| + 2 \geqslant 3$. If $Y_{w_{k_1}} \cup X_{w_{k_1},w_{l_1}}$ is not an independent set of $G$, we may suppose that $x_1y_1 \in E(G)$, as $Y_{w_{k_1}}$ and $X_{w_{k_1},w_{l_1}}$ are both independent sets, and so $|Y_{w_{k_1}}| = 1$ by Fact 15. If $|X_{w_{k_1},w_{l_1}}| = 1$, then $H \cong P_4 \cong W_{2,2}$ (as $x_1y_1 \in E(G)$). If $|X_{w_{k_1},w_{l_1}}| = 2$, then $x_1$ is adjacent to exactly one vertex of $X_{w_{k_1},w_{l_1}}$ by Fact 16. This implies $H \cong K_1 \vee P_4$.

Now, we suppose that $U \neq V(H)$. There exists $z_1 \in V(H) \cap V_2$ such that $z_1 \notin U$ and $z_1$ is not adjacent to some vertex of $U$. We claim that

$$\text{either } z_1 \in X_{w_{k_1},w_{l_2}} \text{ with } l_2 \neq l_1, \text{ or } z_1 \in Y_{w_{l_1}}. \tag{5}$$

Since $z_1 \in V_2 = S_{q-1} \cup S_{q-2}$, we have either $z_1 \in X_{w_{k_2},w_{l_2}}$ or $z_1 \in Y_{w_{l_2}}$, for some vertices $w_{k_2}, w_{l_2} \in V(K_q)$.

If $z_1 \in X_{w_{k_2},w_{l_2}}$, we first show that $|\{k_1, l_1\} \cap \{k_2, l_2\}| = 1$. By way of contradiction and $z_1 \notin X_{w_{k_1},w_{l_1}}$, assume that $|\{k_1, l_1\} \cap \{k_2, l_2\}| = 0$. From Facts 19 and 20, $z_1$ is adjacent to every vertex of $\{w_{k_1}, w_{l_1}\} \cup Y_{w_{k_1}} \cup X_{w_{k_1},w_{l_1}}$, which is impossible. Thus, $|\{k_1, l_1\} \cap \{k_2, l_2\}| = 1$. By Facts 19 and 22(i), we have $X_{w_{k_1},w_{l_1}} = \{y_1\}$, $X_{w_{k_2},w_{l_2}} = \{z_1\}$ and $y_1z_1 \notin E(G)$. Further, from $x_1 \in Y_{w_{k_1}}$ and Fact 22(ii), we get $\{k_1, l_1\} \cap \{k_2, l_2\} = \{k_1\}$.

If $z_1 \in Y_{w_{l_2}}$, then we show that $l_2 = l_1$. As before, assume that $l_2 \neq l_1$. Since $z_1 \notin Y_{w_{k_1}}$, it holds $l_2 \notin \{k_1, l_1\}$. By Fact 20, $z_1$ is adjacent to every vertex of $\{w_{k_1}, w_{l_1}\} \cup X_{w_{k_1},w_{l_1}}$, and thus we may suppose that $x_1z_1 \notin E(G)$ by the choice of $z_1$. But this contradicts Fact 22(ii), and so $l_2 = l_1$. By the former arguments, we have either $z_1 \in X_{w_{k_1},w_{l_2}}$ with $l_2 \neq l_1$ or $z_1 \in Y_{w_{l_1}}$ and (5) holds. Accordingly, we distinguish the following subcases.

*S*ubcase 3.1: $z_1 \in X_{w_{k_1},w_{l_2}}$.
Here we have $X_{w_{k_1},w_{l_1}} = \{y_1\}$, $X_{w_{k_1},w_{l_2}} = \{z_1\}$ and $y_1z_1 \notin E(G)$, by Facts 19 and 22(i).

Since $\{y_1, z_1, w_{k_1}, w_{l_1}, w_{l_2}\} \cup Y_{w_{k_1}} \subseteq V(H)$, there exists $c \geqslant 2$ such that $c$ is as large as possible and $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\} \cup Y_{w_{k_1}} \subseteq V(H)$, where $X_{w_{k_1},w_{s_i}} = \{x_{s_i}\}$ for $1 \leqslant i \leqslant c$. Next, we show that

$$V(H) = \{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\} \cup Y_{w_{k_1}}. \tag{6}$$

Otherwise, there exists a vertex $z_2 \in V(H) \cap V_2$ not belonging to $z_2 \notin \{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\} \cup Y_{w_{k_1}}$ and non-adjacent to any vertex of $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}, w_{k_1}, w_{s_1}, w_{s_2}, \ldots, w_{s_c}\} \cup Y_{w_{k_1}}$. Without loss of generality, we may suppose that $z_2$ is not adjacent to some vertex of $\{x_{s_1}, w_{k_1}, w_{s_1}\} \cup Y_{w_{k_1}}$. By (5) and the symmetry of $z_1$ and $z_2$, either $z_2 \in X_{w_{k_1}, w_h}$ with $h \neq s_1$ or $z_2 \in Y_{w_{s_1}}$ holds.

If $z_2 \in Y_{w_{s_1}}$, then $\min\{|Y_{w_{s_1}}|, |X_{w_{k_1}, w_{s_2}}|, |X_{w_{k_1}, w_{s_1}}|\} \geqslant 1$, contradicting Fact 22(ii). Thus, we have $z_2 \in X_{w_{k_1}, w_h}$ with $h \neq s_1$. Since $z_2 \notin \{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\}$, we have $h \notin \{s_1, s_2, \ldots, s_c\}$, contrary to the choice of $c$. Hence, (6) holds.

From Facts 19 and 21, it holds that $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\}$ and $Y_{w_{k_1}}$ are two independent sets of $G$, respectively. If $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\} \cup Y_{w_{k_1}}$ is also an independent set of $G$, then $H \cong W_{t,c}$, for $t = 1 + |Y_{w_{k_1}}| + c \geqslant 4$. Otherwise, by Facts 15 and 22(iii), we have $Y_{w_{k_1}} = \{x_1\}$ and $x_1$ is adjacent to every vertex of $\{x_{s_1}, x_{s_2}, \ldots, x_{s_c}\}$. This implies $H \cong W_{t,t}$, where $t = c + 1 \geqslant 3$.

*S*ubcase 3.2: $z_1 \in Y_{w_{l_1}}$.
We claim that

$$V(H) = \{w_{k_1}, w_{l_1}\} \cup Y_{w_{k_1}} \cup Y_{w_{l_1}} \cup X_{w_{k_1}, w_{l_1}}. \tag{7}$$

Otherwise, assume that there is a vertex $z_2$ satisfying $z_2 \in V(H) \cap V_2$ and $z_2 \notin \{w_{k_1}, w_{l_1}\} \cup Y_{w_{k_1}} \cup Y_{w_{l_1}} \cup X_{w_{k_1}, w_{l_1}}$, such that $z_2$ is not adjacent to some vertex of $\{w_{k_1}, w_{l_1}\} \cup Y_{w_{k_1}} \cup Y_{w_{l_1}} \cup X_{w_{k_1}, w_{l_1}}$. By the symmetry of $k_1$ and $l_1$, we may suppose that $z_2$ is not adjacent to some vertex of $\{w_{k_1}, w_{l_1}\} \cup Y_{w_{k_1}} \cup X_{w_{k_1}, w_{l_1}}$. From (5) and the symmetry of $z_1$ and $z_2$, we have either $z_2 \in X_{w_{k_1}, w_h}$ with $h \neq l_1$, or $z_2 \in Y_{w_{l_1}}$. Moreover, by the choice of $z_2$, we have $z_2 \in X_{w_{k_1}, w_h}$, with $h \neq l_1$. Then $\min\{|Y_{w_{l_1}}|, |X_{w_{k_1}, w_h}|, |X_{w_{k_1}, w_{l_1}}|\} \geqslant 1$, but this contradicts Fact 22(ii). Hence, (7) is confirmed.

By symmetry, suppose that $1 \leqslant |Y_{w_{k_1}}| \leqslant |Y_{w_{l_1}}|$. Recall that $x_1 \in Y_{w_{k_1}}$, $z_1 \in Y_{w_{l_1}}$ and $y_1 \in X_{w_{k_1}, w_{l_1}}$.

We first suppose that $|Y_{w_{l_1}}| = 1$, that is, $Y_{w_{k_1}} = \{x_1\}$ and $Y_{w_{l_1}} = \{z_1\}$. If $x_1 z_1 \in E(G)$, then $y_1$ is not adjacent to any vertex of $\{x_1, z_1\}$ by Fact 14. By Fact 16, we have $X_{w_{k_1}, w_{l_1}} = \{y_1\}$ and thus $H \cong D_{2,2}$. Otherwise, $x_1 z_1 \notin E(G)$, and then, by Fact 14, $|N_G(y_1) \cap \{x_1, z_1\}| \in \{0, 2\}$ holds. Fact 16 ensures that every vertex of $\{x_1, z_1\}$ is adjacent to the same one vertex of $X_{w_{k_1}, w_{l_1}}$ for the case $|X_{w_{k_1}, w_{l_1}}| = 2$. Since $|X_{w_{k_1}, w_{l_1}}| \leqslant 2$ and $X_{w_{k_1}, w_{l_1}}$ is an independent set of $G$, by employing Fact 16, we arrive at $H \in \{K_1 \vee C_5, K_1 \vee P_4, C_5\}$.

Next, we suppose that $|Y_{w_{l_1}}| \geqslant 2$. By Facts 15 and 17, $Y_{w_{l_1}}$ is an independent set of $G$, and its every vertex is adjacent to every vertex of $Y_{w_{k_1}}$. In addition, $X_{w_{k_1}, w_{l_1}} = \{y_1\}$ and $y_1$ is not adjacent to any vertex of $Y_{w_{l_1}} \cup Y_{w_{k_1}}$. This implies $H \cong D_{a,b}$, where $b = |Y_{w_{l_1}}| + 1 \geqslant 3$ and $a = |Y_{w_{k_1}}| + 1 \geqslant 2$.

Gathering the obtained conclusions, we obtain that $H$ is as in the statement formulation.

Moreover, if $H \cong D_{a,b}$ with $b \geqslant a \geqslant 2$ and $a + b \geqslant 6$, by the former arguments, we deduce that $|Y_{w_{k_1}}| + |Y_{w_{l_1}}| = a + b - 2 \geqslant 4$ and $|X_{w_{k_1}, w_{l_1}}| = 1 \leqslant \min\{|Y_{w_{k_1}}|, |Y_{w_{l_1}}|\}$. By Fact 24, we have $V_1 = \emptyset$, which yields $G \cong G_0$. □

In the next result, we consider one possibility for a component of $\overline{G}_0$; in fact, it is a further characterization of $G_0$.

**Fact 26.** *If $H$ is a non-complete component of $\overline{G}_0$ with independence number $p$, then $P_3 \cup (p-2)K_1 \subset H$.*

*Proof.* Due to Fact 25, we immediately have that the diameter of $H$ is at most 3 and $K_{1,3} \not\subset H$. Suppose that $U = \{v_1, v_2, \dots, v_p\}$ is a maximum independent set of $H$. Since $H$ is non-complete we have $p \geqslant 2$ and $P_3 \subset H$. Hence, the desired conclusion holds for $p = 2$. Suppose that $p \geqslant 3$. If the distance between $v_1$ and $v_2$ is equal to 2, then let $P : v_1 u v_2$ be an induced path of $H$. Since $K_{1,3} \not\subset H$, by taking into account that the diameter of $H$ does not exceed 3, we deduce that $N_H(v_i) \cap V(P) = \emptyset$ holds for $3 \leqslant i \leqslant p$, which implies $P_3 \cup (p-2)K_1 \subset H$.

We next suppose that the distance between $v_1$ and $v_2$ is at least 3. Taking into account the diameter of $H$, we conclude that this distance is exactly 3. Let $P : v_1 u_1 u_2 v_2$ be an induced path of $H$. If $N_H(v_i) \cap V(P) = \emptyset$ holds for $3 \leqslant i \leqslant p$, then we have what we need: $P_3 \cup (p-2)K_1 \subset H$. By the symmetry, we next assume that $N_H(v_3) \cap V(P) \neq \emptyset$. Then $N_H(v_3) \cap V(P) = \{u_1, u_2\}$ holds, as $K_{1,3} \not\subset H$ and $U$ is an independent set. In other words, $V(P) \cup \{v_3\}$ induces $W_{3,2}$. Since $P_3 \cup K_1 \subset W_{3,2} \subset H$, we further suppose that $p \geqslant 4$. If there exists a vertex of $\{v_4, v_5, \dots, v_p\}$, say $v_4$, such that $N_H(v_4) \cap V(P) \neq \emptyset$, then $N_H(v_4) \cap V(P) = \{u_1, u_2\}$, as before. But this implies that $\{v_1, u_1, v_3, v_4\}$ induces $K_{1,3}$, which is eliminated at the beginning of this proof. Hence, $N_H(v_i) \cap V(P) = \emptyset$ holds for $4 \leqslant i \leqslant p$, which implies $W_{3,2} \cup (p-3)K_1 \subset H$. The desired result follows from $P_3 \cup (p-2)K_1 \subset W_{3,2} \cup (p-3)K_1$. $\square$

We proceed to consider the existence of $D_{a,b}$ in the collection of components of $\overline{G}_0$. First, we limit the parameters $a$ and $b$ as follows.

**Fact 27.** *Let $D_{a,b}$ be a component of $\overline{G}_0$. If $b \geqslant a \geqslant 4$ or $b \geqslant 6$ and $a = 3$, then either $\overline{G} \subset D_{4,4} \cup H$ for $H \in \{3K_1 \cup \overline{W}_{3,3}, 4K_1 \cup C_5, W_{6,6}, 5K_1 \cup K_4\}$ or $\overline{G} \subset D_{3,6} \cup H$ for $H \in \{3K_1 \cup \overline{W}_{3,3}, 4K_1 \cup C_5, W_{6,6}\}$.*

*Proof.* By Fact 25, we have $G_0 \cong G$ for $a + b \geqslant 6$. Since $6K_1 \cup D_{4,5}$ and $6K_1 \cup D_{3,7}$ are not $\sigma$-graphs, we have $6K_1 \cup D_{4,5} \not\subset \overline{G}$ and $6K_1 \cup D_{3,7} \not\subset \overline{G}$. As $\omega(G) \geqslant 8$, for $b \geqslant a \geqslant 4$ or $b \geqslant 6$ and $a = 3$, we have either $a = b = 4$ or $b = 6$ and $a = 3$. Let $F \in \{D_{4,4}, D_{3,6}\}$ be a component of $\overline{G}$. We shall distinguish these possibilities for $F$ in the end of the proof; at this moment it is sufficient to take that $F$ is one of these graphs. Since $7K_1 \cup F \not\subset \overline{G}$ and $|V(F) \cap V(K_q)| = 2$, we have $q = 8$.

We claim that $\overline{G}$ contains at most two non-trivial components. Otherwise, suppose that $G_1$, $G_2$ are non-trivial components distinct from $F$. Since $F \cup 2P_3 \cup 2K_1 \not\subset \overline{G}$, $F \cup P_3 \cup K_2 \cup 3K_1 \not\subset \overline{G}$ and $\omega(G) = 8$, Fact 26 implies that both $G_1$ and $G_2$ are complete. Consequently, $F \cup 2K_2 \cup 4K_1 \subset \overline{G}$, which contradicts $\lambda_2(G) \leqslant \sigma$. Thus, $\overline{G}$ contains at most two non-trivial components, as claimed.

Assume that $\overline{G}$ is not an induced subgraph of $F \cup H$ for $H \in \{3K_1 \cup \overline{W}_{3,3}, 4K_1 \cup C_5, W_{6,6}\}$. Then, $\overline{G}$ contains exactly two non-trivial components, as $\omega(G) = 8$ and $F \cup$

$6K_1 \subset F \cup W_{6,6}$. Let $G_1$ be the other non-trivial component. By the assumption, we have $G_1 \notin \{\overline{W}_{3,3}, C_5, W_{6,6}\}$. Since $F \cup D_{2,2} \not\subset \overline{G}$ and $F \cup 4K_1 \cup (K_1 \vee P_3) \not\subset \overline{G}$, we have $G_1 \cong W_{c,j}$ for $0 \leqslant j \leqslant c$ by Fact 25, as $G_1 \notin \{\overline{W}_{3,3}, C_5\}$.

We claim that $j \in \{0, c-1, c\}$. Otherwise, there would be $\omega(\overline{W}_{c,j}) = j+1$ and $F \cup W_{3,1} \cup 4K_1 \subset F \cup W_{c,j} \cup (5-j)K_1 \cong \overline{G}$, as $\omega(G) = 8$ and $W_{3,1} \cup (j-1)K_1 \subset W_{c,j}$. This contradicts $W_{3,1} \cup 4K_1 \cup F \not\subset \overline{G}$. Thus, $j \in \{0, c-1, c\}$, and this implies that $\overline{G}$ is isomorphic to either $F \cup K_c \cup 5K_1$ for $c \geqslant 2$, or $F \cup W_{c,c-1} \cup (6-c)K_1$ for $2 \leqslant c \leqslant 6$, or $F \cup W_{c,c} \cup (6-c)K_1$ for $2 \leqslant c \leqslant 6$. Combining this with $\overline{G} \not\subset F \cup W_{6,6}$, we obtain $H \cong K_c$ for $c \geqslant 2$.

It remains to consider $F$. If $F \cong D_{4,4}$, then $\overline{G} \subset D_{4,4} \cup 5K_1 \cup K_4$, as $D_{4,4} \cup 5K_1 \cup K_5 \not\subset \overline{G}$. Otherwise, we have $F \cong D_{3,6}$. Since $D_{3,6} \cup 5K_1 \cup K_3 \not\subset \overline{G}$, it holds $\overline{G} \subset D_{3,6} \cup 5K_1 \cup K_2 \subset D_{3,6} \cup 4K_1 \cup C_5$, which contradicts the assumption of $\overline{G}$. This completes the proof. □

Now, we set $a = 2$.

**Fact 28.** *Let $D_{2,b}$ be a component of $\overline{G}_0$. If $b \geqslant 19$, then $\overline{G}$ is an induced subgraph of some graph of $\{D_{2,27} \cup 6K_1, D_{2,21} \cup W_{6,6}, D_{2,21} \cup \overline{W}_{3,3} \cup 3K_1, D_{2,21} \cup C_5 \cup 4K_1, D_{2,19} \cup K_4 \cup 5K_1\}$.*

*Proof.* We denote $H \cong D_{2,b}$. By Fact 25, we have $G_0 \cong G$ even for $b \geqslant 4$. Since $b \geqslant 19$ and $7K_1 \cup D_{2,14} \not\subset \overline{G}$, we have $q = 8$.

We claim that $\overline{G}$ contains at most two non-trivial components. Otherwise, suppose that $G_1, G_2$ are two non-trivial components distinct from $H$. Note that $D_{2,19} \cup 2K_2 \cup 4K_1 \not\subset \overline{G}$. Since $D_{2,19} \cup 2P_3 \cup 2K_1 \not\subset \overline{G}$, $D_{2,19} \cup P_3 \cup K_2 \cup 3K_1 \not\subset \overline{G}$ and $\omega(G) = 8$, $G_1$ and $G_2$ are complete by Fact 26, which leads to the impossible scenario $D_{2,19} \cup 2K_2 \cup 4K_1 \subset \overline{G}$.

If $D_{2,t}$ is the unique non-trivial component of $\overline{G}$, then $\overline{G} \subset D_{2,27} \cup 6K_1$, as $D_{2,28} \cup 6K_1 \not\subset \overline{G}$. Next, let $G_1$ be the other (apart from $H$) non-trivial component. Since $D_{2,12} \cup D_{2,2} \not\subset \overline{G}$ and $D_{2,13} \cup G_1 \cup 4K_1 \not\subset \overline{G}$ for $G_1 \in \{W_{3,1}, K_1 \vee P_3\}$, by Fact 25 we have

$$G_1 \in \{C_5, \overline{W}_{3,3}, W_{c,c-1}, W_{c,c}, K_c\}, \quad \text{for a proper } c \geqslant 2. \tag{8}$$

For $G_1 \cong \overline{W}_{3,3}$, since $D_{2,22} \cup \overline{W}_{3,3} \cup 3K_1 \not\subset \overline{G}$, we obtain $b \leqslant 21$. It follows that $\overline{G} \subset D_{2,21} \cup \overline{W}_{3,3} \cup 3K_1$. For $G_1 \cong C_5$, from $D_{2,22} \cup C_5 \cup 4K_1 \not\subset \overline{G}$, we deduce $b \leqslant 21$, and thus $\overline{G} \subset D_{2,21} \cup C_5 \cup 4K_1$. For $G_1 \in \{W_{c,c}, W_{c,c-1}\}$ for some $c \geqslant 2$, we have $b \leqslant 21$, as follows from $D_{2,22} \cup P_3 \cup 4K_1 \not\subset \overline{G}$ and Fact 26. This implies $\overline{G} \subset D_{2,21} \cup W_{6,6}$.

The remaining case of (8) is $G_1 \cong K_c$, for $c \geqslant 2$. Since $D_{2,22} \cup K_2 \cup 5K_1 \not\subset \overline{G}$, we have $b \leqslant 21$. When $20 \leqslant b \leqslant 21$, from $D_{2,20} \cup K_3 \cup 5K_1 \not\subset \overline{G}$ we have $c = 2$. Then $\overline{G} \subset D_{2,21} \cup K_2 \cup 5K_1 \subset D_{2,21} \cup W_{6,6}$. When $b = 19$, from $D_{2,19} \cup K_5 \cup 5K_1 \not\subset \overline{G}$ we have $c \leqslant 4$. Then $\overline{G} \subset D_{2,19} \cup K_4 \cup 5K_1$.

Gathering the obtained possibilities for $\overline{G}$, we arrive at the desired result. □

We are in position to prove Theorem 7.

*Proof of Theorem 7.* The sufficiency for (i) follows from Corollary 6, whereas for (ii)–(iv) it is confirmed by direct computation.

We proceed with the necessity, i.e., we assume that $G$ is as in the statement of the theorem. As noted before Fact 25, we have $G \cong G_0 \vee (K_h|W_{h,h})$ with $V(G_0) = V(G) \setminus$

$N_{K_q}[V_1]$. If $\overline{G}_0$ does not contain $D_{a,b}$ as a component, then each component of $\overline{G}_0$ is an induced subgraph of either $W_{c,c}$, or $\overline{W}_{3,3}$, or $K_1 \vee C_5$, as follows from Fact 25. Therefore, there exists a sufficiently large $s \geqslant h$, such that $G \subset H_s \vee (K_s|W_{s,s})$ where $\overline{H}_s = s((K_1 \vee C_5) \cup \overline{W}_{3,3} \cup W_{s,s})$. In other words, $G$ is an induced subgraph of some graph of $\mathcal{D}$, as in (i).

We next suppose that $D_{a,b}$ is a component of $\overline{G}_0$. For $b \geqslant 19$ and $a = 2$, or $b \geqslant a \geqslant 4$, or $b \geqslant 6$ and $a = 3$, Facts 27 and 28 lead to (ii)–(iv). If either $a = 2$ and $4 \leqslant b \leqslant 18$, or $a = 3 \leqslant b \leqslant 5$, then $G \cong G_0$ by Fact 25. This implies that $\overline{G}$ contains a special component, which contradicts the assumption of this theorem. The remaining possibility is $a = 2 \leqslant b \leqslant 3$. Once again, by Fact 25, each component of $\overline{G}_0$ is an induced subgraph of either $W_{c,c}$, or $D_{2,3}$, or $\overline{W}_{3,3}$, or $K_1 \vee C_5$. As before, this leads to the conclusion that $G$ is as in (i). The proof is completed. □

**Corollary 29** (**Necessity in Theorem 2**). *Let $G$ be a $\sigma$-graph satisfying either*

*(i)* $\omega(G) \geqslant 54$ *or*

*(ii)* $\omega(G) \geqslant 8$ *and $G$ has a pendant vertex.*

*Then $G$ is an induced subgraph of some graph of* $\mathcal{D} \cup \{\overline{D}_{2,4} \vee K_{52}\}$.

*Proof.* We first suppose that $G$ satisfies (ii). If $\overline{G}_0$ contains $D_{a,b}$ as a component for $a + b \geqslant 6$, then Fact 25 yields $V_1 = \emptyset$, and thus $V(G) = V_2 \cup V(K_q)$. Since every vertex of $V_2$ is adjacent to at least two vertices of $K_q$, $G$ has no pendant vertex, which contradicts the assumption of this corollary. It remains to consider the case in which $D_{a,b}$, for $a+b \geqslant 6$, does not appear as a component of $\overline{G}_0$. Then, by virtue of Fact 25, for every component $H$ of $\overline{G}_0$ we have either $H \cong W_{t,j}$ with $0 \leqslant j \leqslant t$ and $t \geqslant 1$, or $H \cong D_{a,b}$ with $2 \leqslant a \leqslant b \leqslant 3$, or $H \in \{K_1 \vee P_3, K_1 \vee P_4, K_1 \vee C_5, C_5, \overline{W}_{3,3}\}$. In the first case we have $H \subset W_{t,t}$; in the second case, we have $H \subset D_{2,3}$; in the third case, we have $H \subset K_1 \vee C_5$ or $H \subset \overline{W}_{3,3}$. Summa summarum, $G$ is an induced subgraph of some graph of $\mathcal{D}$.

In the remainder of the proof we suppose that $G$ satisfies item (i) of the corollary. If $\overline{G}$ has no any special component, then one of items (i)–(iv) of Theorem 7 holds. Moreover, if $G$ is as in Theorem 7(ii)–(iv), then $\omega(G) < 54$ (even more, it is at most 8) contrary to the assumption of the corollary. The remaining possibility (that is Theorem 7(i)) states that $G$ is an induced subgraph of some graph of $\mathcal{D}$.

Otherwise, $\overline{G}$ contains a special component $D_{a,b}$. If $b \geqslant a \geqslant 3$, then $17K_1 \cup D_{3,3} \subset \overline{G}$ (as $\omega(G) \geqslant 54$ and $\omega(\overline{D}_{a,b}) = 2$), but this is impossible since $\lambda_2(K_{17} \vee \overline{D}_{3,3}) > 0.619 > \sigma$. Hence, $b \geqslant 4$ and $a = 2$. Together with $53K_1 \cup D_{2,4} \not\subset \overline{G}$, this implies $\omega(G) = 54$. Next, from $\omega(G) = 54$ and $19K_1 \cup D_{2,5} \not\subset \overline{G}$, we obtain $b = 4$. For $(a, b) = (2, 4)$, Fact 25 implies $G \cong G_0$. Thus, we only need to determine the structure of $G_0$.

We claim that $D_{2,4}$ is the unique non-trivial component of $\overline{G} \cong \overline{G}_0$. Otherwise, suppose that $H$ is another non-trivial component. If $H$ is a complete graph, then we have $F_1 \cong D_{2,4} \cup K_2 \cup 51K_1 \subset \overline{G}$, since $\omega(G) = 54$ and $\omega(\overline{D}_{2,4} \vee \overline{H}) = 3$. If $H$ is non-complete, applying Fact 26 to $H$ and $\overline{G}$ (in the role of $\overline{G}_0$), we arrive at $F_2 \cong D_{2,4} \cup P_3 \cup 50K_1 \subset \overline{G}$, since $\omega(G) = 54$ and $\omega(\overline{D_{2,4} \cup P_3}) = 4$. Both possibilities are eliminated because $\overline{F_1}$ and $\overline{F_2}$ are not $\sigma$-graphs as confirmed directly.

Since $D_{2,4}$ is the unique non-trivial component of $\overline{G}$, we have $\overline{G} \cong D_{2,4} \cup kK_1$. From $\omega(G) = 54$ and $\omega(\overline{D}_{2,4}) = 2$, we obtain $k = 52$, and thus $\overline{G} \cong D_{2,4} \cup 52K_1$. The proof is completed. □

## 5 Concluding remarks

Here we mention some details observed during the research, give directions for future works and make more comparisons with the existing results. To make the reading easier, for some notions introduced in the previous sections we refer to the first occurrence in the text.

We have determined the $\sigma$-graphs whose clique number is at least 54. Simultaneously, we have established all $\sigma$-graphs whose clique number is at least 8, under the additional assumption that they contain at least one pendant vertex. At the first glance, someone would expect a long list of the resulting graphs, but it occurs that they fall into the induced subgraphs of exactly two structured types (if we consider the class $\mathcal{D}$ as one of them and the single graph $\overline{D}_{2,4} \vee K_{52}$ as the other one).

The next natural step would be to consider the remaining $\sigma$-graphs whose clique number is at least 8, where 'remaining' refers to those without pendant vertices. Theorem 7 can be seen as a first step in this direction, as it provides the desired $\sigma$-graphs $G$ under the caveat that $\overline{G}$ has no any special component, where a special component is defined upon the same theorem. Next, if $\overline{G}$ has a special component, then by Fact 25, we have $G \cong G_0$; again, $G_0$ is defined upon the same fact. With slight modifications in the proofs of Facts 27 and 28, one may extend the result of Theorem 7. For example, the case in which $\overline{G}$ has the special component $D_{2,18}$ can be easily resolved, since there $\omega(G) = 8$ must hold (provided by $7K_1 \cup D_{2,14} \not\subset \overline{G}$). However, it seems that extending Facts 27 and 28, and consequently Theorem 2, to cover all the remaining special components would be a rather difficult task.

Besides, $\sigma$-graphs $G$ satisfying $8 \leqslant \omega(G) \leqslant 53$ can be considered from the following perspective. By Theorem 7, we may suppose that $\overline{G}$ contains at least one special component (say $D_{a,b}$). In this case, by Fact 25, we have $V_1 = \emptyset$ and $G \cong G_0$. Since $D_{2,4} \cup D_{3,3} \not\subset \overline{G}$, $2D_{2,4} \not\subset \overline{G}$ and $2D_{3,3} \not\subset \overline{G}$, we deduce that $\overline{G}$ contains exactly one special component. Now, from Fact 25, we obtain that $\overline{G}$ is isomorphic to

$$D_{a,b} \cup a_1(K_1 \vee P_3) \cup a_2(K_1 \vee P_4) \cup a_3(K_1 \vee C_5) \cup a_4 C_5 \cup a_5 \overline{W_{3,3}} \cup a_6 D_{2,2} \cup a_7 D_{2,3} \cup \bigcup_{i=1}^{a_8} W_{t_i,j_i} \cup \bigcup_{i=1}^{a_9} K_{s_i},$$

where $a_i \geqslant 0$ for $1 \leqslant i \leqslant 9$, $t_i \geqslant j_i \geqslant 1$ for $0 \leqslant i \leqslant a_8$, $s_i \geqslant 1$ for $0 \leqslant i \leqslant a_9$ and $D_{a,b}$ is a special component of $\overline{G}$ (with a convention that $\bigcup_{i=1}^{0} F_i$ is an empty graph).

On the basis of Fact 25, we determine connected graphs with given clique number and $\lambda_2 \leqslant \frac{1}{2}$.

**Proposition 30.** *Let $G$ be a connected graph with clique number at least 19. Then $\lambda_2(G) < \frac{1}{2}$ if and only if each component of $\overline{G}$ is either $K_r$ with $r \geqslant 1$, or $W_{2,1}$, or $W_{3,1}$.*

*Proof.* By using the Courant-Weyl Inequalities, we get $\lambda_2(G)+\lambda_{\min}(\overline{G})\leqslant\lambda_2(K_n)=-1$. One can easily check that $\lambda_{\min}(H)>-\frac{3}{2}$ holds for $H\in\{K_r,W_{2,1},W_{3,1}\}$. Therefore, $\lambda_2(G)<\frac{1}{2}$ holds for every graph given in the statement of this proposition.

For the necessity, we always suppose that $\omega(G)=q\geqslant 19$ and $K_q$ is a fixed maximum clique of $G$.

Since $\lambda_2(G)<\frac{1}{2}<\frac{\sqrt{5}-1}{2}$, Fact 25 remains valid for $G$. Since the set $V_1$, of vertices in $V(G)\setminus V(K_q)$ having exactly one neighbour in $K_q$, is empty (as follows directly by taking into account $q\geqslant 19$ and $\lambda_2<\frac{1}{2}$), we may reformulate this fact as follows.

**Fact 17′**. *For any component $H$ of $\overline{G}$, there is either $H\cong W_{t,j}$ with $0\leqslant j\leqslant t$ and $t\geqslant 1$, or $H\cong D_{a,b}$ with $b\geqslant a\geqslant 2$, or*

$$H\in\{K_1\vee P_3,K_1\vee P_4,K_1\vee C_5,C_5,\overline{W}_{3,3}\}.$$

We discuss the possibilities for $H$. If $H\cong K_r$ for some $r\geqslant 1$, we are done. Next, we assume that $H$ is not a complete graph. By Fact 17′, either $H\cong W_{t,j}$ with $1\leqslant j\leqslant t$ and $t\geqslant 2$ (note that $W_{t,0}\cong K_t$ and $W_{1,1}\cong K_2$), or $H\cong D_{a,b}$ with $b\geqslant a\geqslant 2$, or $H\in\{K_1\vee P_3,K_1\vee P_4,K_1\vee C_5,C_5,\overline{W}_{3,3}\}$.

Since $\lambda_2(\overline{P_4})=\frac{\sqrt{5}-1}{2}>\frac{1}{2}$, we obtain $P_4\not\subset\overline{G}$. So, the case $H\in\{K_1\vee P_4,K_1\vee C_5,C_5,\overline{W}_{3,3}\}$ cannot occur. Observing that $\omega(\overline{D_{a,b}})=\omega(\overline{K_1\vee P_3})=2$ and taking into account the restriction $q\geqslant 19$, we obtain $H\cong W_{t,j}$ with $1\leqslant j\leqslant t$ and $t\geqslant 2$, as $\lambda_2(\overline{D_{2,2}\cup 5K_1})>\frac{1}{2}$ and $\lambda_2(\overline{(K_1\vee P_3)\cup 2K_1})>\frac{1}{2}$.

Since $P_4\not\subset\overline{G}$, we have $j=1$ and $t\geqslant 2$. We also have $\omega(\overline{W_{t,1}})=2$, along with $\lambda_2(\overline{W_{4,1}\cup 17K_1})>\frac{1}{2}$. Hence, $t\leqslant 3$, which implies $H\cong W_{3,1}$ for $t=3$ and $H\cong W_{2,1}$ for $t=2$. □


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