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Research Article

Signed Toral Tessellations Whose Spectrum Consists of Exactly Two Symmetric Eigenvalues

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Abstract

Signed graphs with exactly two eigenvalues constitute a rich and extensively studied class, yet they remain far from fully classified. Many structural properties are known and numerous families have been constructed, and any new non-trivial construction continues to offer notable progress. In this paper, the known family of signed graphs with eigenvalues 2 and -2, previously recognized as the 4-regular toral tessellations, is extended by the introduction of an infinite family of signed graphs whose spectra consist solely of two symmetric eigenvalues $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, where λ is an unbounded integer. Furthermore, a complete characterization of imposed signed toral tessellations is provided; one of its consequences is that every such graph necessarily has even order and even vertex degree.

Keywords: signed graph; repeating pattern; orthogonally similar matrices; Kronecker product.

2020 Mathematics Subject Classification: 05C22, 05C50.

1. Introduction

A *signed graph* $\dot{G}$ has edges declared either *positive* or *negative*. Precisely, $\dot{G} = (G, \sigma)$, where G is a finite simple undirected graph, called the *underlying graph*, and σ is a *sign function* assigning either $+1$ or -1 to each edge of G. The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard adjacency matrix of G by reversing the sign of every entry that corresponds to a negative edge. The eigenvalues and the spectrum of $A_{\dot{G}}$ are considered as the *eigenvalues* and the *spectrum* of $\dot{G}$.

Henceforth, when we say that a signed graph has two eigenvalues, we mean that it has exactly two distinct eigenvalues. In contrast to the unsigned case, where two eigenvalues occur if and only if the connected graph in question is complete, signed graphs with two eigenvalues form a rich, well-studied, yet far from fully characterised class. These signed graphs are necessarily regular, moreover strongly regular in the sense of [7]. Their connection to systems of lines in Euclidean spaces that are pairwise either orthogonal or meet at a fixed angle is noted in [9]. All signed graphs with two eigenvalues and vertex degree at most five are known [4, 5, 9, 10]. Signed line graphs with the same spectral property are determined in [9], where the definition of a signed line graph can also be found. Further results concern signed graphs with two eigenvalues, each greater than or equal to -2 [8]. Other studies address sporadic constructions based on particular graph operations or matrix products [2, 4, 6, 9]. In summary, many properties of signed graphs with two eigenvalues have been established, and numerous families of such graphs have been constructed. At this stage, the field appears largely explored, and any new non-trivial construction of such signed graphs would constitute a significant contribution.

This is precisely the aim of the present paper. More specifically, we extend the construction of signed graphs with eigenvalues 2 and -2 presented in [5] to obtain an infinite family of signed graphs with two symmetric eigenvalues $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, where λ is an unbounded integer. Our starting point is the family of 4-regular signed graphs illustrated in Figure 1.1, introduced in the aforementioned reference. The authors referred to this family as a 4-regular toral tessellation. The term is intuitive, as from a topological perspective the graph can be viewed as a covering of the torus, exhibiting a repeating pattern based on the 4-vertex induced subgraph indicated in the figure. It is not difficult to see that $A^2_{\dot{T}_{2k}} = 4I$, which yields the desired spectrum.

In this paper, we generalize the pattern to an arbitrary even number of vertices, split into two equal parts that together form a bipartite signed graph. As in the previous construction, this pattern is repeated cyclically. We determine when the resulting tessellation has exactly two symmetric eigenvalues and provide a complete characterization of the corresponding signed graphs. Several illustrative examples are also included. The results are framed within a broader context by examining the key properties of the corresponding pattern.

The main results are presented in the next section, followed by a separate section containing examples.

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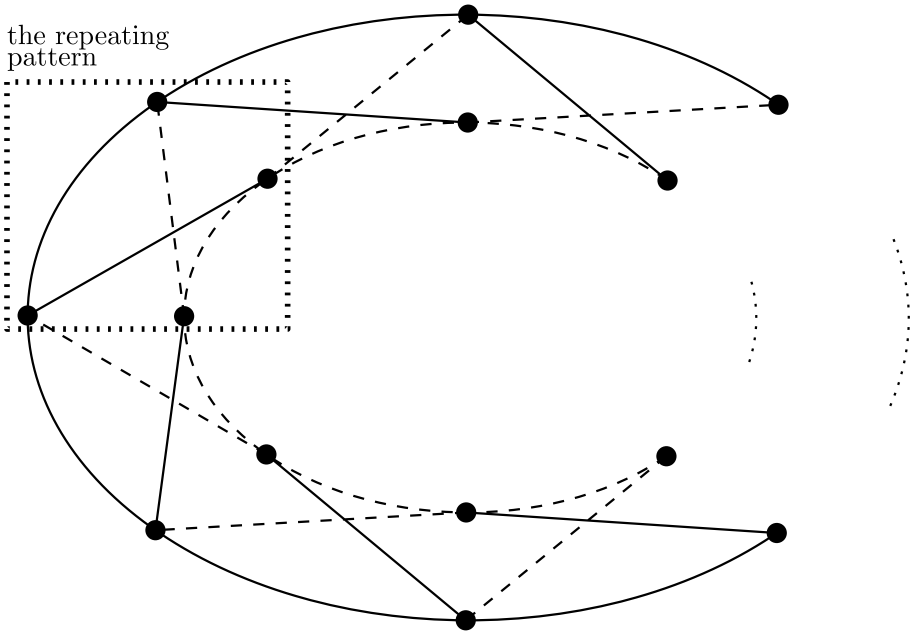

Figure 1.1: The 4-regular toral tessellations $\dot{T}_{2k}$, $k \geq 3$. Negative edges are dashed. The repeating pattern is framed.

2. Pattern construction and its properties

We begin with a more general framework involving a graph constructed from an n-vertex cycle. Each vertex of the cycle is replaced by a cell consisting of mutually non-adjacent vertices, while each edge is replaced by a prescribed set of edges connecting the vertices of two neighbouring cells. A corresponding signed graph is then obtained by assigning a specified edge signature. Its adjacency matrix takes the form

$$B = \begin{pmatrix} O & A_1 & O & \cdots & O & A_n^\intercal \\ A_1^\intercal & O & A_2 & \cdots & O & O \\ O & A_2^\intercal & O & \cdots & O & O \\ \vdots & & & \ddots & & \vdots \\ O & O & O & \cdots & O & A_{n-1} \\ A_n & O & O & \cdots & A_{n-1}^\intercal & O \end{pmatrix}. \tag{1}$$

Theorem 2.1. *The identity $B^2 = 2pI$ holds if and only if the following two families of conditions are satisfied (indices are* mod n*):*

(i) $A_iA_i^\intercal + A_{i-1}^\intercal A_{i-1} = 2pI$;

(ii) $A_iA_{i+1} = O$.

Proof. By definition $(B^2)_{i,j} = \sum_{m=1}^{n} B_{i,m}B_{m,j}$. Since each row i of B has non-zero blocks only at columns $i-1$ and $i+1$, we have

$$(B^2)_{i,j} = B_{i,i-1}B_{i-1,j} + B_{i,i+1}B_{i+1,j} = A_{i-1}^\intercal B_{i-1,j} + A_iB_{i+1,j}.$$

Now note that each $B_{i,j}$ is non-zero only when $j = i \pm 1$. Hence, only the following block positions of B^2 can be non-zero:

$$(B^2)_{i,i} = A_{i-1}^\intercal A_{i-1} + A_iA_i^\intercal,$$

$$(B^2)_{i,i+2} = A_iA_{i+1},$$

$$(B^2)_{i,i-2} = A_{i-1}^\intercal A_{i-2}^\intercal.$$

For necessity, if $B^2 = 2pI$, then all off-diagonal blocks of B^2 vanish. Thus, for all i,

$$A_iA_{i+1} = O \quad \text{and} \quad A_{i-1}^\intercal A_{i-2}^\intercal = O.$$

Since the two off-diagonal families are transposes of one another (and hence equivalent by reindexing), we may retain only $A_iA_{i+1} = O$. The diagonal blocks of B^2 yield

$$A_iA_i^\intercal + A_{i-1}^\intercal A_{i-1} = 2pI.$$

Conversely, assume the stated two families of relations hold. Then, for all i,

$$(B^2)_{i,i+2} = A_iA_{i+1} = 0, \quad (B^2)_{i,i-2} = (A_{i-2}A_{i-1})^\intercal = 0,$$

and the diagonal blocks satisfy $(B^2)_{i,i} = 2pI$. Hence, all off-diagonal blocks of B^2 vanish, while each diagonal block equals $2pI$, and therefore $B^2 = 2pI$. □

Evidently, B features as the adjacency matrix of a signed graph with exactly two eigenvalues: $\sqrt{2p}$ and $-\sqrt{2p}$. An alternative proof of the previous theorem uses NEPS products of signed graphs [11, Subsection 2.3.2], extending a method previously applied to a similar structure in the unsigned case [1].

We now restrict ourselves to the pattern announced in the previous section. Precisely, for $n \geq 1$, we are interested in $n \times n$ real matrices A with entries in $\{1, 0, -1\}$ and with exactly p non-zero entries in each row and each column (so p is an integer $0 \leq p \leq n$) that satisfy

$$A^2 = O, \quad AA^T + A^TA = 2pI_n. \tag{2}$$

Here are some basic properties of A.

Proposition 2.1. *Let $A \in \mathbb{R}^{n\times n}$. If $A^2 = O$ then every eigenvalue of A is 0. If A satisfies (2), then $p \geq 0$. If, in addition, $p = 0$ then $A = O$.*

Proof. The first part of the statement is a well-known property of nilpotent matrices. A short proof reads: $A\mathrm{x} = \lambda\mathrm{x}$ implies $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x} = \mathbf{0}$, and so $\lambda = 0$ as $\mathbf{x} \neq \mathbf{0}$.

Suppose that A satisfies both identities of (2). Since A is square matrix, both $AA^\intercal$ and $A^\intercal A$ are positive semidefinite, and thus their sum is also positive semidefinite. Since it equals $2pI$, we must have $p \geq 0$. If $p = 0$, then

$$\|A\mathbf{x}\|^2 + \|A^\intercal\mathbf{x}\|^2 = 0,$$

for $\mathbf{x} \in \mathbb{R}^n$. Hence, $A = O$. □

We proceed with more sophisticated properties.

Theorem 2.2. *Let $A \in \mathbb{R}^{n\times n}$ satisfy (2) with $p > 0$. Then:*

(a) *n is even and* $\mathrm{rank}(A) = n/2$.

(b) ***(Orthogonal canonical form)*** *There exists an orthogonal matrix Q such that*

$$Q^\intercal AQ = \sqrt{2p}\bigoplus_{j=1}^{n/2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Conversely, every matrix orthogonally similar to this block form satisfies (2).

(c) *The singular values of A are $\sqrt{2p}$ and 0, each with multiplicity $n/2$.*

Proof. (a): Let $r = \mathrm{rank}(A)$. Since $A^2 = 0$, it holds $\mathrm{Im}(A) \subseteq \ker(A)$, implying $r \leq n - r$ and therefore $r \leq n/2$.

For $\mathbf{x} \in \ker(A)$, the latter identity of (2) gives

$$2p\|\mathbf{x}\|^2 = \|A\mathbf{x}\|^2 + \|A^\intercal\mathbf{x}\|^2 = \|A^\intercal\mathbf{x}\|^2.$$

Since $p > 0$, $A^\intercal$ is injective on $\ker(A)$, so $n - r \leq r$, giving $r \geq n/2$. Summa summarum, $r = n/2$ and n is even.

(b): We give the canonical-form construction. Since $A^2 = O$ and $\mathrm{rank}(A) = r = n/2$, choose a complementary subspace V to $U = \mathrm{Im}(A)$, so that

$$\mathbb{R}^n = U \oplus V, \quad \dim(U) = \dim(V) = r.$$

With respect to bases of U and V, A transforms into the block form

$$\begin{pmatrix} O & S \\ O & O \end{pmatrix},$$

where S represents the restriction of $A\colon V \to U$. From the latter identity of (2), we obtain

$$SS^\intercal = S^\intercal S = 2p\, I_r.$$

Thus, $S = \sqrt{2p}\, S'$ for some orthogonal matrix S'. Finally, an orthogonal change of basis within U and V transforms S' into the identity. Combining this with the bases of U and V, we obtain the canonical form

$$Q^\intercal A Q = \sqrt{2p} \bigoplus_{j=1}^{r} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The converse statement is a consequence of a direct algebraic computation.

(c): Using (b), we write

$$A = Q \left(\sqrt{2p} \bigoplus_{j=1}^{n/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) Q^\intercal, \tag{3}$$

which holds since Q is orthogonal. Singular values are invariant under orthogonal similarity. For one block

$$\sqrt{2p} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the singular values are $\sqrt{2p}$ and 0. With $n/2$ blocks, the full matrix has singular values $\sqrt{2p}$ and 0, and they are equal in multiplicity. □

The next theorem extends part (b) of the previous one, providing the main characterization of A. The remaining parts, together with Proposition 2.1 in its entirety, can be viewed as refinements that give further properties of the same matrix.

Theorem 2.3. *All real $n \times n$ matrices satisfying (2) belong to a single orthogonal similarity class. In particular, every such matrix is orthogonally similar to*

$$\sqrt{2p} \bigoplus_{j=1}^{n/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

They are nilpotent of index 2 and have Jordan canonical form consisting of exactly $n/2$ Jordan blocks of size 2.

The proof is a rephrasing of the argument used in part (b). The latter theorem characterizes all matrices A that produce the desired pattern, i.e., those for which the matrix B defined in (1) satisfies $B^2 = 2pI$. Consequently, such matrices A are given by (3) for an appropriate choice of an orthogonal matrix Q that yields $\{1, 0, -1\}$-entries. In the next section, we provide several explicit constructions.

3. Examples

Both examples presented in this section actually yield iterative constructions that generate infinite families of matrices satisfying (2).

Example 3.1. *Let A_0 be an $n_0 \times n_0$ $\{1, 0, -1\}$-matrix satisfying (2) with parameter p_0. Let W be an $m \times m$ weighing matrix with $WW^\intercal = wI_m$ (entries in $\{1, 0, -1\}$ and w non-zeros per row/column). Then*

$$A_1 = A_0 \otimes W$$

is the $n_0 m \times n_0 m$ $\{1, 0, -1\}$-matrix. In addition, we have $A_1^2 = (A_0 \otimes W)^2 = A_0^2 \otimes W^2 = O$, along with

$$\begin{aligned} A_1 A_1^\intercal + A_1^\intercal A_1 &= (A_0 \otimes W)(A_0^\intercal \otimes W^\intercal) + (A_0^\intercal \otimes W^\intercal)(A_0 \otimes W) \\ &= (A_0 A_0^\intercal) \otimes (WW^\intercal) + (A_0^\intercal A_0) \otimes (W^\intercal W) \\ &= (A_0 A_0^\intercal + A_0^\intercal A_0) \otimes (wI_m) \\ &= (2p_0 I_{n_0}) \otimes (wI_m) \\ &= 2p_0 w\, I_{n_0 m}. \end{aligned}$$

Hence, A satisfies (2) with parameter $p = p_0 w$.

Accordingly, we have established an iterative construction:

$$\begin{cases} A_0, & \text{an arbitrary } \{1,0,-1\}\text{-matrix satisfying (2)},\\ A_{i+1} = A_i \otimes W, & i \geq 1. \end{cases}$$

Of course, W does not need to be fixed in each iteration. It is worth mentioning that many results on weighing matrices, including the complete classification of those with $w \leq 5$, can be found in [3]. A canonical choice for A_0 is

$$A_0 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

the matrix corresponding to the pattern in Figure 1.1.

Example 3.2. *Let A_0 be a $k \times k$ $\{1,0,-1\}$-matrix satisfying (2). For any $n \geq 2$ define the $nk \times nk$ block-cyclic matrix*

$$A_1 = \begin{pmatrix} 0 & A_0 & 0 & \cdots & 0 \\ 0 & 0 & A_0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ A_0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

A direct calculation shows $A_1^2 = 0$, along with

$$A_1A_1^\intercal + A_1^\intercal A_1 = \operatorname{diag}\big(A_0A_0^\intercal + A_0^\intercal A_0, \ldots, A_0A_0^\intercal + A_0^\intercal A_0\big) = I_k \otimes (A_0A_0^\intercal + A_0^\intercal A_0) = 2p\, I_{nk}.$$

Moreover, A_1 has entries in $\{1,0,-1\}$ and exactly p non-zeros per row/column (coming from the single block A_0 per wrap). For $i \geq 2$, A_i is constructed as before.

It is straightforward to verify that every matrix constructed in the previous examples has the form (3). We conclude this section, and the paper as a whole, with some additional structural remarks on A:

- The matrices $P_L := \frac{1}{2p}AA^\intercal$ and $P_R := \frac{1}{2p}A^\intercal A$ are orthogonal projections of rank $n/2$. They project onto the left and the right singular subspaces respectively.
- The polar decomposition $A = U|A|$ has U a partial isometry with initial space the column space and final space the row space of A.
- Interpreting A as the adjacency matrix of a bipartite regular signed graph (rows vs columns), the condition $A^2 = O$ forbids length-2 closed walks; combinatorially this imposes restrictions on the support pattern.

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