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Unbalanced signed graphs with extremal spectral radius or index

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Abstract
Let $\dot{G} = (G, \sigma)$ be a signed graph, and let $\rho(\dot{G})$ (resp. $\lambda_1(\dot{G})$) denote the spectral radius (resp. the index) of the adjacency matrix $A_{\dot{G}}$. In this paper we detect the signed graphs achieving the minimum spectral radius $m(\mathcal{SR}_n)$, the maximum spectral radius $M(\mathcal{SR}_n)$, the minimum index $m(\mathcal{I}_n)$ and the maximum index $M(\mathcal{I}_n)$ in the set $\mathcal{U}_n$ of all unbalanced connected signed graphs with $n \geq 3$ vertices. From the explicit computation of the four extremal values it turns out that the difference $m(\mathcal{SR}_n) - m(\mathcal{I}_n)$ for $n \geq 8$ strictly increases with n and tends to 1, whereas $M(\mathcal{SR}_n) - M(\mathcal{I}_n)$ strictly decreases and tends to 0.

Keywords Signed graph · Unbalanced graph · Switching equivalence · Index · Spectral radius

Mathematics Subject Classification 05C22 · 05C50 · 15A18

1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a simple graph and $\sigma : E \longrightarrow \{+1, -1\}$ is the sign function (or the *signature*) defined on the edge set $E = E(G)$. The unsigned graph G is called the *underlying graph* of $\dot{G}$. The *order* and the *size* of $\dot{G}$ are the order and the size of its underlying graph. The *sign* of a cycle $\dot{C}$ in $\dot{G}$ is defined as

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$\operatorname{sign}(\dot{C}) = \prod_{e \in \dot{C}} \sigma(e)$. A cycle is called *positive* (resp. *negative*) if $\operatorname{sign}(\dot{C})$ is 1 (resp. -1). A signed graph is *balanced* if no negative cycles exist; otherwise it is *unbalanced*. We interpret an unsigned graph as a signed graph without negative edges. The *negation* $-\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$.

The adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard adjacency matrix A_G (of the underlying graph) by replacing 1 by -1 whenever the corresponding edge is negative. By the *spectrum* of $\dot{G}$, we mean the spectrum of $A_{\dot{G}}$. Since $A_{\dot{G}}$ is symmetric, its eigenvalues $\lambda_1(\dot{G}) \geq \lambda_2(\dot{G}) \geq \cdots \geq \lambda_n(\dot{G})$ are real. Moreover, since the trace of $A_{\dot{G}}$ is equal to zero, we have $\lambda_1(\dot{G})\lambda_n(\dot{G}) \leq 0$, with equality if and only if $\dot{G}$ is empty (without edges). The *index* of $\dot{G}$ is simply the largest eigenvalue $\lambda_1(\dot{G})$, whereas the *spectral radius* $\rho(\dot{G})$ of $\dot{G}$ is the largest absolute value of its eigenvalues, i.e.

$$\rho(\dot{G}) = \max\{\lambda_1(\dot{G}),\ -\lambda_n(\dot{G})\}.$$

Since, in general, $A_{\dot{G}}$ is not similar to a non-negative matrix, it may happen that $-\lambda_n(\dot{G}) > \lambda_1(\dot{G})$. By virtue of Theorems 1.2 and 1.5 of Stevanović (2015) (the latter one is the celebrated Perron–Frobenius Theorem), this surely occurs for $-G$ whenever it is non-bipartite. On the contrary, if $\dot{G}$ is balanced, then $\rho(\dot{G}) = \lambda_1(\dot{G})$ (see (Stevanović 2015,Theorem 1.5)).

The last decades have seen a growing interest for the ‘spectral’ sub-branch of extremal graph theory. It essentially consists in identifying those objects which are extremal with respect to a fixed spectral parameter within a given class of graphs. In particular, some extremal problems have been solved in the context of signed graphs. For instance, in Koledin and Stanić (2017) the authors studied connected signed graphs of fixed order, size and number of negative edges that maximize the index. In the wake of that paper, signed graph maximizing the index in suitable subsets of complete signed graphs have been studied in Akbari et al. (2020); Ghorbani and Majidi (2021). Let $\mathfrak{U}_n$ (resp. $\mathfrak{B}_n$) denote the class of unbalanced unicyclic (resp. bicyclic) signed graphs of order n. (We recall that the definition of an unicyclic signed graph does not deviate from the same definition in the domain of unsigned graphs, so it is a connected signed graph whose order is equal to its size. Bicyclic signed graphs are defined accordingly.) Akbari et al. (2019) determined signed graphs with extremal index in $\mathfrak{U}_n$. Some of the same authors studied in Souri et al. (2020) signed graphs achieving the maximum index among all graphs in $\mathfrak{U}_n$ of fixed girth. The first five largest indices among graphs in $\mathfrak{B}_n$ with $n \geq 36$ are detected by He et al. (2021). Signed graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with extremal spectral radius were identified in Belardo et al. (2021). Finally, extremal graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with respect to the least Laplacian eigenvalue were studied in Belardo and Zhou (2016) and Belardo et al. (2018), respectively.

Throughout the paper we write $\mathcal{G}_n$ to denote the set of connected signed graphs of order n, and we use $\mathcal{B}_n$ to denote the subset of connected balanced signed graphs. The complementary set $\mathcal{U}_n = \mathcal{G}_n \backslash \mathcal{B}_n$ of connected unbalanced signed graphs is clearly non-empty provided $n \geq 3$.

We positively address the following problem: for every $n \geq 3$, determine the signed graph(s) in $\mathcal{U}_n$ attaining the minimum or the maximum of the following two sets

$$\mathcal{SR}_n = \{\rho(\dot{G}) \mid \dot{G} \in \mathcal{U}_n\} \quad \text{and} \quad \mathcal{I}_n = \{\lambda_1(\dot{G}) \mid \dot{G} \in \mathcal{U}_n\}. \tag{1}$$

As we shall see in Sect. 4, the minimizers and the maximizers with respect to the index are complete signed graphs (with a suitable distribution of negative edges).

The remainder of the paper is structured as follows. Section 2 contains some terminology and notation along with some preliminary results. Sections 3 and 4 are respectively devoted to seek out the signed graphs achieving the extremal values in $\mathcal{SR}_n$ and $\mathcal{I}_n$ (the efforts required to face the maximal index have been slightly harder). Section 5 contains some concluding remarks and a comparison to the existing results concerning balanced signed graphs with extremal index.

2 Basic tools and preliminaries

For a signed graph $\dot{G} = (G, \sigma)$ and a function $\theta\colon V(G) \longrightarrow \{+1, -1\}$, we can build a new signed graph $\dot{G}^{\theta} = (G, \sigma^{\theta})$, where $\sigma^{\theta}(e) = \theta(v_i)\sigma(e)\theta(v_j)$ for each edge $e = v_iv_j \in E(G)$. The signed graphs $\dot{G}$ and $\dot{G}^{\theta}$ are said to be *switching equivalent*; they share the same spectrum and the same set of positive cycles. It follows by (Zaslavsky 1989,Lemma 5.3) that $\dot{G}$ is balanced if and only if $\dot{G}$ is switching equivalent to its underlying graph. Moreover, the signed graphs $\dot{G}$ and $\dot{H}$ are said to be *switching isomorphic* if $\dot{G}$ is isomorphic to a signed graph that is switching equivalent to $\dot{H}$. In this case we write $\dot{G} \backsimeq \dot{H}$.

If $\dot{G}$ is switching isomorphic to $\dot{H}^{\theta}$ for a suitable

$$\theta\colon V(H) = \{w_1, w_2, \ldots, w_n\} \longrightarrow \{+1, -1\},$$

then $A_{\dot{H}^{\theta}} = P^{-1}D^{-1}A_{\dot{G}}DP$, where P is an appropriate permutation (0, 1)-matrix and D is the diagonal matrix diag $(\theta(w_1), \theta(w_2), \ldots, \theta(w_n))$. Since, by the previous consideration, all unbalanced cycles of fixed order are spectrally indistinguishable, with a slight abuse of notation we denote by $\dot{C}_n^-$ any unbalanced cycle of order $n \geq 3$.

We say that $\dot{H} = (H, \tau)$ is an (induced) subgraph of $\dot{G} = (G, \sigma)$ if H is an (induced) subgraph of G and $\tau = \sigma|_H$. We write $\dot{G} - v$ (resp. $\dot{G} - e$) to denote the signed graph obtained from $\dot{G}$ by deleting the vertex v (resp. the edge e). We recall that the Cauchy's Interlacing Theorem holds for every Hermitian matrix (see (Cvetković et al. 1995,Theorem 0.10)), and therefore the eigenvalues of $\dot{G} - v$ interlace the eigenvalues of $\dot{G}$.

We note that if $\dot{G}$ is bipartite, then the index $\lambda_1(\dot{G})$ is equal to the spectral radius $\rho(\dot{G})$. This result follows from the fact that $\dot{G}$ and $-\dot{G}$ are switching equivalent, and thus $\lambda_1(\dot{G}) = \lambda_1(-\dot{G}) = -\lambda_n(\dot{G})$.

The following result reproduces (Akbari et al. 2019,Theorem 2.5); it can be alternatively proved on the basis of (Stanić 2019a,Theorem 3.1).

Theorem 2.1 *For a non-empty signed graph $\dot{G}$ of order n,*

$$\lambda_n(-G) \leq \lambda_n(\dot{G}) < \lambda_1(\dot{G}) \leq \lambda_1(G). \tag{2}$$

Here is a simple corollary.

Corollary 2.2 *For a signed graph $\dot{G}$, $\rho(\dot{G}) \leq \rho(G)$.*

Proof It is sufficient to realize that $\lambda_n(-G)$ can be replaced by $-\lambda_1(G)$ in (2). □

The next result characterizes connected signed graphs for which the third inequality of (2) is actually the equality.

Proposition 2.3 (Stanić 2019b,Lemma 2.1) *Let $\dot{G}$ be a connected signed graph. Then $\lambda_1(\dot{G}) = \lambda_1(G)$ if and only if $\dot{G}$ is balanced.*

Let $\lambda_{i_1}(\dot{G}) > \lambda_{i_2}(\dot{G}) > \cdots > \lambda_{i_k}(\dot{G})$ be all distinct eigenvalues of $\dot{G}$. Throughout the paper we adopt the following notation to denote the spectrum of (the adjacency matrix of) $\dot{G}$:

$$\mathrm{Spec}(\dot{G}) = \big[\lambda_{i_k}^{(n_k)}, \lambda_{i_{k-1}}^{(n_{k-1})}, \ldots, \lambda_{i_1}^{(n_1)}\big],$$

where the exponents stand for the multiplicities. Although the following lemma is known to scholars since at least fifty years (see the historical remarks in (Stanić 2015,Section 2.1)), we provide in any case a quick proof for sake of completeness.

Lemma 2.4 *For a connected unsigned graph G with n vertices, $\lambda_1(G) \leq n-1$, with equality if and only if $G = K_n$.*

Proof Use the fact that $\lambda_1(K_n) = n - 1$ and the fact (proved, for instance, in Stanić 2015,Corollary 1.5)) that the index strictly increases when we add an edge between two non-adjacent vertices of a non-complete connected graph. □

We now provide a similar result in the framework of signed graphs.

Proposition 2.5 *For a signed graph $\dot{G}$ with n vertices, $\lambda_1(\dot{G}) \leq n - 1$, with equality if and only if $\dot{G}$ is balanced and complete.*

Proof By Theorem 2.1 and Lemma 2.4 we obtain

$$\lambda_1(\dot{G}) \leq \lambda_1(G) \leq n - 1. \tag{3}$$

Assume now that $\lambda_1(\dot{G}) = n - 1$. We deduce from (3) that $\lambda_1(\dot{G}) = \lambda_1(G) = n - 1$. The first equality and Proposition 2.3 imply that $\dot{G}$ is balanced. The second equality and Lemma 2.4 imply that the underlying graph G is complete, and the proof is over. □

3 Unbalanced signed graphs with extremal spectral radius

Taking into account the spectrum of the complete graph, we immediately get

$$\mathrm{Spec}(-K_n) = [1 - n, 1^{(n-1)}].$$

The next theorem shows that an unbalanced connected graph of order $n \geq 3$ maximizes the spectral radius if and only if it is switching equivalent to $-K_n$.

Theorem 3.1 *For $n \geq 3$, let $\mathcal{U}_n$ be the set of unbalanced connected graphs of order n, and let $M(\mathcal{SR}_n)$ denote the maximum of the set $\mathcal{SR}_n = \{\rho(\dot{G}) \mid \dot{G} \in \mathcal{U}_n\}$. Then $M(\mathcal{SR}_n) = n - 1$ and*

$$\{\dot{G} \in \mathcal{U}_n \mid \rho(\dot{G}) = n - 1\} = \{\dot{G} \in \mathcal{U}_n \mid \dot{G} \backsimeq -K_n\}. \tag{4}$$

Proof By using, in turn, Corollary 2.2, the Perron–Froebenius Theorem and Lemma 2.4, we see that

$$\rho(\dot{G}) \leq \rho(G) = \lambda_1(G) \leq n - 1.$$

Hence, $M(\mathcal{SR}_n) = n - 1$.

Assume that for a fixed $\dot{G} \in \mathcal{U}_n$ we have $\rho(\dot{G}) = n - 1$. Since $\dot{G}$ is unbalanced, Proposition 2.5 ensures that $-\lambda_n(\dot{G}) = n-1$, and then we obtain $\lambda_1(-\dot{G}) = n-1$. We now use again Proposition 2.5 to deduce that this is possible only if $-\dot{G}$ switches to K_n or, equivalently, if $\dot{G} \backsimeq -K_n$ as claimed. □

To face signed graphs that minimize the spectral radius we need the following two lemmas. Signed graphs with comparatively small spectral radius were studied in McKee and Smyth (2007). We frame one of their more striking results in the first lemma.

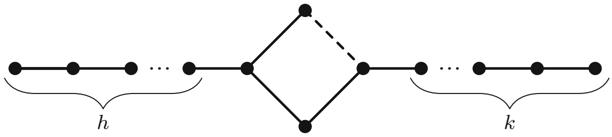

Fig. 1 The signed graph $\dot{Q}_{h,k}$. Here and in the forthcoming figures, negative edges are depicted by dashed lines

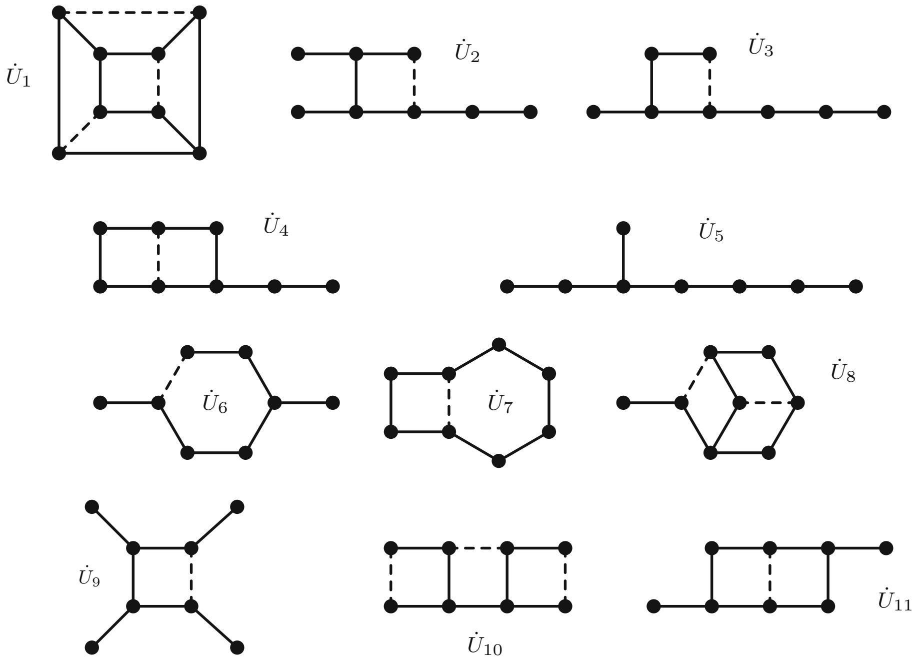

Fig. 2 The eleven signed graphs $\dot{U}_1, \dot{U}_2, \dots, \dot{U}_{11}$ whose spectra lie in $(-2,\ 2)$. All of them have 8 vertices

Lemma 3.2 (McKee and Smyth 2007,Theorem 4) *Up to switching isomorphism, connected signed graphs having all their eigenvalues in* $(-2,\ 2)$ *are the induced subgraphs of:*

(i) the unbalanced cycle $\dot{C}_{2k}^{-}$*,*
(ii) the signed graph $\dot{Q}_{h,k}$*, illustrated in Fig. 1 or*
(iii) the eleven sporadic examples $\dot{U}_1, \dot{U}_2, \dots, \dot{U}_{11}$ *with* 8 *vertices illustrated in Fig. 2.*

In what follows we give more details related to the signed graphs having the form as in Fig. 1. In particular, we point out their significance in the study of signed graphs that minimize the spectral radius in the set of unbalanced unicyclic signed graphs. We resume the notation from the opening section.

Lemma 3.3 *Let* $\dot{Q}_{h,k}$ *be the signed graph illustrated in Fig. 1, and let* $\mathfrak{U}_n$ *be the set of unbalanced unicyclic signed graphs of order* $n \geq 3$*. The following statements hold.*

(i) For $h \geq 0$*,* $\dot{Q}_{h,h}$ *shares the spectrum with the unbalanced cycle* $\dot{C}_{2h+4}^{-}$*.*
(ii) Signed graphs minimizing the spectral radius in $\mathfrak{U}_n$ *are switching equivalent either to* $\dot{Q}_{\lceil \frac{n-4}{2} \rceil, \lfloor \frac{n-4}{2} \rfloor}$ *or (if n is even) to* $\dot{C}_n^{-}$*.*
(iii) For $h \geq 1$*,* $\rho(\dot{Q}_{h-1,h}) = \rho(\dot{Q}_{h,h}) = 2\cos(\pi/(2h+4))$*.*

Table 1 The values of $m(\mathcal{SR}_n)$ for $n \geq 3$

n	3	4	5	6	7	8	≥ 9
$m(\mathcal{SR}_n)$	2	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$2\cos\frac{\pi}{2\lceil\frac{n}{2}\rceil}$

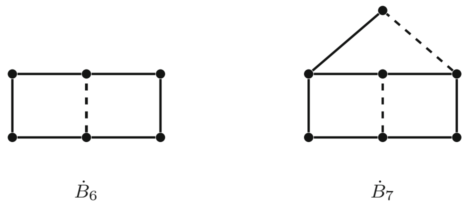

Fig. 3 Minimizers of the spectral radius in $\mathcal{U}_6$ and $\mathcal{U}_7$

Proof Part (i) comes from (Akbari et al. 2018,Theorem 21) and (ii) rephrases (Belardo et al. 2021,Theorem 4.2).

For (iii), note that the underlying graphs of both $\dot{Q}_{h-1,h}$ and $\dot{Q}_{h,h}$ are bipartite, and so $\rho(\dot{Q}_{h,h}) = \lambda_1(\dot{Q}_{h,h})$ and $\rho(\dot{Q}_{h-1,h}) = \lambda_1(\dot{Q}_{h-1,h})$. By Part (ii) and (Akbari et al. 2018,Table 1) we immediately get that the index of $\dot{Q}_{h,h}$ is $2\cos(\pi/(2h+4))$ and has multiplicity 2. Now, observe that $\dot{Q}_{h-1,h}$ is the induced subgraph of $\dot{Q}_{h,h}$ obtained by deleting one of its pendant vertices. Hence, the equality $\lambda_1(\dot{Q}_{h,h}) = \lambda_1(\dot{Q}_{h-1,h}) = 2\cos(\pi/(2h+4))$ is a consequence of the Cauchy's eigenvalue interlacing. □

Remark 3.4 Part (iii) of the previous lemma can be alternatively deduced from (Akbari et al. 2018,Theorem 3.6) for $h > 1$, and by a direct computation for $h = 1$.

Now we are in position to prove the main result of this section. (The notation for two bicyclic signed graphs ($\dot{B}_6$ and $\dot{B}_7$) is transferred from Belardo et al. (2021)).

Theorem 3.5 *For $n \geq 3$, let $m(\mathcal{SR}_n)$ denote the minimum of the set $\mathcal{SR}_n = \{\rho(\dot{G}) \mid \dot{G} \in \mathcal{U}_n\}$. Then the values of $m(\mathcal{SR}_n)$ are given in Table 1. Moreover, up to switching isomorphism, the signed graphs in $\mathcal{U}_n$ minimizing the spectral radius are the following:*

- *the unbalanced cycle $\dot{C}_n^-$ for $n \in \{3, 4\}$;*
- *$\dot{Q}_{0,1}$ for $n = 5$;*
- *$\dot{C}_6^-$, $\dot{Q}_{1,1}$, and the signed graph $\dot{B}_6$ illustrated in Fig. 3 for $n = 6$;*
- *the signed graph $\dot{B}_7$ illustrated in Fig. 3 for $n = 7$;*
- *the signed graph $\dot{U}_1$ illustrated in Fig. 2 for $n = 8$;*
- *$\dot{Q}_{\lceil\frac{n-4}{2}\rceil,\lfloor\frac{n-4}{2}\rfloor}$ or (if n is even) $\dot{C}_n^-$ for $n \geq 9$.*

Proof Since $\mathcal{U}_3$ just contains unbalanced triangles, $m(\mathcal{SR}_3) = \rho(\dot{C}_3^-) = 2$. For $n \geq 3$, Lemma 3.2 says that $m(\mathcal{SR}_n) < 2$. Once we input Lemma 3.3(ii), we conclude that minimizers of the spectral radius in $\mathcal{U}_n$ must be sought among $\dot{C}_{2k}^-$ for $k \geq 2$, $\dot{Q}_{h-1,h}$, $\dot{Q}_{h-1,h}$ for $h \geq 1$ and the induced subgraphs of $\dot{U}_i$ with $1 \leq i \leq 11$ and $i \neq 5$ (see Fig. 2; $\dot{U}_5$ is eliminated since it is balanced). Therefore, for $n \in \{4, 5\} \cup \{2k-1 \mid k \geq 5\}$, up to switching isomorphism, there is exactly one candidate of order n: $\dot{C}_4^-$ for $n = 4$, $\dot{Q}_{0,1}$ for $n = 5$ and $\dot{Q}_{k-3,k-2}$ for $n = 2k-1$ with $k \geq 5$. For $n = 2k$ and $k \geq 5$ the two possible candidates ($\dot{C}_{2k}^-$ and $\dot{Q}_{k-2,k-2}$) share the same spectrum, hence they are both minimizers.

The minimizers for $6 \leq n \leq 8$ must be searched among (a) the unbalanced cycles, (b) the signed graphs $\dot{Q}_{1,1}$, $\dot{Q}_{1,2}$ and $\dot{Q}_{2,2}$, (c) the minimizers of the spectral radius among unicyclic and bicyclic unbalanced graphs determined in Belardo et al. (2021) and (d) the induced subgraphs of $\dot{U}_1, \dot{U}_8, \dot{U}_{10}$ having at least three independent cycles.

The statement now comes from a direct analysis of the spectral radii of this finite set of signed graphs. In particular, it turns out that

$$\rho(\dot{Q}_{0,1}) = \rho(\dot{B}_6) = \rho(\dot{B}_7) = \rho(\dot{U}_1) = \sqrt{3}.$$

whereas the formula

$$\rho\big(\dot{Q}_{\lceil \frac{n-4}{2} \rceil, \lfloor \frac{n-4}{2} \rfloor}\big) = 2\cos\frac{\pi}{2\lceil \frac{n}{2} \rceil},$$

is equivalent to Lemma 3.3(iii). □

4 Unbalanced signed graphs with extremal index

Complete signed graphs with n vertices $v_1, v_2, \ldots, v_n$ and just one negative edge are all pairwise switching isomorphic. In order to fix ideas, we select among them $\dot{K}_n^M = (K_n, \sigma^M)$, where

$$\sigma^M : e \in V(K_n) \longmapsto \begin{cases} -1 & \text{if } e = v_1 v_2, \\ 1 & \text{otherwise.} \end{cases}$$

Our aim is to show that a signed graph maximizes the index in $\mathcal{U}_n$ if and only it is switching isomorphic to $\dot{K}_n^M$. To prove this we need a sequence of auxiliary lemmas. We first compute the spectrum of $\dot{K}_n^M$.

Lemma 4.1 *For $n \geq 3$, the spectrum of $\dot{K}_n^M$ is*

$$\operatorname{Spec}(\dot{K}_n^M) = \big[\, a_n, (-1)^{(n-3)}, 1, b_n \,\big],$$

where

$$a_n = \frac{n - \sqrt{(n-2)(n+6)}}{2} - 2 \quad \text{and} \quad b_n = \frac{n + \sqrt{(n-2)(n+6)}}{2} - 2.$$

Moreover,

$$\begin{cases} a_n \leq -2 \\ n - 2 \leq b_n < n - 1 \end{cases} \qquad \text{(the equalities hold only for } n = 3\text{)}. \tag{5}$$

Proof A direct computation shows that $\operatorname{Spec}(\dot{K}_3^M) = [\, a_3, 1, b_3 \,]$, with $a_3 = -2$ and $b_3 = 1$. For the remainder of this proof, we assume $n \geq 4$.

Let I_n be the $n \times n$ identity matrix, and let $J_{m \times n}$ be the $m \times n$ all-1 matrix. After setting $\tilde{J} = J_{2 \times (n-2)}$, we observe that

$$A_{\dot{K}_n^M} = \left(\begin{array}{cc|c} 0 & -1 & \tilde{J} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{\tilde{J}^\intercal} & A_{K_{n-2}} \end{array}\right)$$

Since the four blocks have constant row sums, by (Cvetković et al. 1995,Theorem 0.12) the multiset $\operatorname{Spec}(\dot{K}_n^M)$ contains the eigenvalues of

$$B_n = \begin{pmatrix} -1 & n-2 \\ 2 & n-3 \end{pmatrix},$$

the matrix obtained from $A_{\dot{K}_n^M}$ by replacing each block with the sum of its rows. Now, $\mathrm{Spec}(B_n) = [a_n, b_n]$, where a_n, b_n are given in the formulation of the statement. Elementary algebraic computation shows that, for $n \geq 4$, $a_n < -2$ and the numbers $n-1-b_n$ (resp. $b_n - (n-2)$) form a strictly decreasing (resp. increasing) sequence taking values in $(0, 1)$, which proves (5).

The proof ends once we note that the multiplicity of -1 in $\mathrm{Spec}(\dot{K}_n^M)$ is $n-3$, the first three rows of $A_{\dot{K}_n^M} + I_n$ being independent, and $1 \in \mathrm{Spec}(\dot{K}_n^M)$, since the first two columns of $A_{\dot{K}_n^M} - I_n$ are equal. □

We now consider a particular eigenvector afforded by b_n.

Lemma 4.2 *With the notation of the previous lemma, for $n \geq 4$, the eigenspace of b_n contains the positive eigenvector*

$$\mathbf{y}^{\mathsf{T}} = \left(-\frac{a_n+1}{2}, -\frac{a_n+1}{2}, \underbrace{1, 1, \ldots, 1}_{n-2}\right). \tag{6}$$

Proof Note that the first two coordinates are indeed positive, since $a_n < -2$. An algebraic manipulation shows that $A_{\dot{K}_n^M}\mathbf{y} = b_n\mathbf{y}$ is equivalent to

$$a_n + b_n = n - 4 \quad \text{and} \quad a_n b_n = 7 - 3n,$$

and these two equalities actually hold since a_n and b_n are the roots of

$$\det(xI_n - B_n) = x^2 - (n-4)x + 7 - 3n,$$

the characteristic polynomial of B_n. □

For the remainder of this section, we fix an edge $e \in E(K_n) \setminus \{v_1v_2\}$ and take into account $\dot{K}_n^M - e$, the subgraph of $\dot{K}_n^M$ obtained by deleting e. By definition, $\sigma^M(e) = 1$.

Lemma 4.3 *For $n \geq 3$, we have*

$$\lambda_1(\dot{K}_n^M - e) > n - 3.$$

Proof The statement is trivially true for $n = 3$ and follows from a direct inspection for $n = 4$. Let $n \geq 5$. Since $e \neq v_1v_2$, one of its ending vertices, say u, is not in the subset $\{v_1, v_2\}$. It follows that $\dot{K}_n^M - e - u$ is isomorphic to $\dot{K}_{n-1}^M$. Now, the eigenvalue interlacing, together with inequalities of (5), gives

$$\lambda_1(\dot{K}_n^M - e) \geq \lambda_1(\dot{K}_{n-1}^M) > n - 3,$$

as claimed. □

Now we show that, for $n \geq 5$, $\lambda_1(\dot{K}_n^M - e)$ is afforded by a positive eigenvector.

Lemma 4.4 *For $n \geq 5$, the eigenspace of $\lambda_1(\dot{K}_n^M - e)$ contains an eigenvector with all positive coordinates.*

Proof To ease language we abbreviate e to uv and $\lambda_1(\dot{K}_n^M - e)$ to λ. There are two possible isomorphic types for $\dot{K}_n^M - e$ depending on whether the intersection $\{u, v\} \cap \{v_1, v_2\}$ is empty or not. These two cases are dealt separately.

Case 1: $\{u, v\} \cap \{v_1, v_2\} = \emptyset$. Since the underlying graph of $\dot{K}_n^M$ is complete, it is not restrictive to assume that $e = v_{n-1}v_n$. A λ-eigenvector $\mathbf{x}$ has the following form:

$$\mathbf{x}^\mathsf{T} = (a, a, \underbrace{b, b, \ldots, b}_{n-4}, c, c),$$

where a, b and c satisfy the following eigenvalue equations:

$$\begin{aligned} \lambda a &= -a + (n-4)b + 2c, \\ \lambda b &= \ 2a + (n-5)b + 2c, \qquad (7) \\ \lambda c &= \ 2a + (n-4)b. \qquad (8) \end{aligned}$$

After subtracting (8) from (7), a straightforward algebraic manipulation leads to

$$a = \frac{\lambda + 2}{\lambda + 3}c \quad \text{and} \quad b = \frac{\lambda + 2}{\lambda + 1}c.$$

Clearly, we have $\lambda > -1$, and therefore the components a, b and c all share the same sign since $\mathbf{x} \neq \mathbf{0}$. It follows that one of $\mathbf{x}$ and $-\mathbf{x}$ is all-positive.
Case 2: $\{u, v\} \cap \{v_1, v_2\} = \{v_2\}$. Recall that $\sigma^M(v_1v_2) = -1$. Up to replacing $\dot{K}_n^M - e$ with an isomorphic copy, we can assume that $e = v_2v_3$. This time, a λ-eigenvector $\mathbf{x}$ has the form:

$$\mathbf{x}^\mathsf{T} = (a, b, c, \underbrace{d, d, \ldots, d}_{n-3}),$$

and eigenvalue equations read as follows:

$$\begin{aligned} \lambda a &= -b + c + (n-3)d, \\ \lambda b &= -a + (n-3)d, \\ \lambda c &= a + (n-3)d, \\ \lambda d &= a + b + c + (n-4)d. \end{aligned}$$

By solving this system, we arrive at

$$b = \frac{(\lambda+1)(\lambda-2)}{\lambda^2}a, \quad c = \frac{(\lambda-1)(\lambda+2)}{\lambda^2}a, \quad d = \frac{\lambda^2 + 2\lambda - 4}{\lambda^2}a.$$

Since we are assuming that $n \geq 5$, Lemma 4.3 ensures that $\lambda > 2$. Hence, if a is positive, so are also b, c and d, and we are done. □

We proceed with the following lemma.

Lemma 4.5 *Every signed graph $\dot{G} \in \mathcal{U}_n$ is a subgraph of some unbalanced complete signed graph $\dot{K}$ (depending on $\dot{G}$) such that $\lambda_1(\dot{G}) \leq \lambda_1(\dot{K})$.*

Proof If $\dot{G}$ is complete there is nothing to prove. If the size of $\dot{G}$ is $m < \binom{n}{2}$, then we can build a sequence of nested signed graphs

$$\dot{G} = \dot{G}_0 \subset \dot{G}_1 \subset \cdots \subset \dot{G}_{\binom{n}{2}-m}$$

such that for $1 \leq k \leq \binom{n}{2} - m$: $\dot{G}_{k-1}$ is an edge-deleted subgraph of $\dot{G}_k$, the size of $\dot{G}_k$ is $m + k$ and $\lambda_1(\dot{G}_{k-1}) \leq \lambda_1(\dot{G}_k)$. Indeed, if $\mathbf{x}$ is an eigenvector associated with $\lambda_1(\dot{G}_{k-1})$ and $\dot{G}_k$ is obtained by adding an edge uw to $\dot{G}_{k-1}$, then by virtue of the Rayleigh principle we have

$$\lambda_1(\dot{G}_k) \geq \mathbf{x}^\mathsf{T} A_{\dot{G}_k}\mathbf{x} = \lambda_1(\dot{G}_{k-1}) + 2\sigma(uw)x_u x_w.$$

Obviously, with an appropriate choice for the signature $\sigma(uw)$ we get $\lambda_1(\dot{G}_k) \geq \lambda_1(\dot{G}_{k-1})$.

Due to the previous construction, $\dot{G}_{\binom{n}{2}-m}$ is complete and unbalanced, since it contains $\dot{G}$ as a subgraph, and $\lambda_1(\dot{G}) \leq \lambda_1\left(\dot{G}_{\binom{n}{2}-m}\right)$. □

From Lemma 4.5 we deduce that there exists at least one unbalanced complete graph whose index is $M(\mathcal{I}_n) = \max\{\lambda_1(\dot{G}) \mid \dot{G} \in \mathcal{U}_n\}$. The next proposition shows that every complete unbalanced graph maximizing the index is switching isomorphic to $\dot{K}_n^M$. We shall use this result in the proof of the forthcoming Theorem 4.7 in order to prove that, apart such complete signed graphs, there are no other unbalanced maximizers.

Proposition 4.6 *If $\dot{K}' = (K_n, \sigma')$ is an unbalanced complete signed graph of order $n \geq 3$ such that $\lambda_1(\dot{K}') = M(\mathcal{I}_n)$, then $\dot{K}'$ is switching isomorphic to $\dot{K}_n^M$.*

Proof For $n = 3$ the statement trivially holds since all unbalanced graphs in $\mathcal{U}_3$ are switching equivalent to $\dot{K}_3^M$. Let now $n \geq 4$. By (Stanić 2018,Lemma 1), there exists a (necessarily unbalanced) complete signed graph $\dot{K}'' = (K_n, \sigma'')$ such that:

(i) $\dot{K}'' \backsimeq \dot{K}'$ and
(ii) there is a non-negative $\lambda_1(\dot{K}'')$-eigenvector $\mathbf{x}$ (of $A_{\dot{K}''}$).

Since $\dot{K}''$ has at least one negative edge, up to replacing it with a switching isomorphic signed graph, we can assume that $\sigma''(v_1v_2) = -1$. Using the Rayleigh principle, we obtain

$$\lambda_1(\dot{K}'') = \mathbf{x}^\mathsf{T} A_{\dot{K}''}\mathbf{x} \quad \text{and} \quad \lambda_1(\dot{K}_n^M) \geq \mathbf{x}^\mathsf{T} A_{\dot{K}_n^M}\mathbf{x}.$$

Therefore,

$$\begin{aligned} 0 \leq M(\mathcal{I}_n) - \lambda_1(\dot{K}_n^M) &= \lambda_1(\dot{K}'') - \lambda_1(\dot{K}_n^M) \\ &\leq \mathbf{x}^\mathsf{T} A_{\dot{K}''}\mathbf{x} - \mathbf{x}^\mathsf{T} A_{\dot{K}_n^M}\mathbf{x} \\ &= \mathbf{x}^\mathsf{T}\big(A_{\dot{K}''} - A_{\dot{K}_n^M}\big)\mathbf{x} \\ &\leq 0, \end{aligned} \tag{9}$$

where the last inequality follows from the fact that the matrix $A_{\dot{K}''} - A_{\dot{K}_n^M}$ is non-positive.

From (9) we infer that $\lambda_1(\dot{K}_n^M) = M(\mathcal{I}_n)$ and $\lambda_1(\dot{K}_n^M) = \mathbf{x}^\mathsf{T} A_{\dot{K}_n^M}\mathbf{x}$. This is possible only if $\mathbf{x}$ is also a $\lambda_1(\dot{K}_n^M)$-eigenvector of $A_{\dot{K}_n^M}$. By Lemmas 4.1 and 4.2, the vector $\mathbf{x}$ is positive. In fact, the multiplicity of $\lambda_1(\dot{K}_n^M) = b_n$ is 1, and $\mathbf{x}$ is proportional to the positive vector $\mathbf{y}$ of (6). With this information at hand, the equality $\mathbf{x}^\mathsf{T}\left(A_{\dot{K}''} - A_{\dot{K}_n^M}\right)\mathbf{x} = 0$ implies $A_{\dot{K}''} = A_{\dot{K}_n^M}$. In other words, $\dot{K}'' = \dot{K}_n^M$. Hence, $\dot{K}' \backsimeq \dot{K}_n^M$, and we are done. □

We are now ready to prove the main result of this section.

Theorem 4.7 *A signed graph $\dot{G} \in \mathcal{U}_n$ such that $\lambda_1(\dot{G}) = M(\mathcal{I}_n)$ is switching isomorphic to $\dot{K}_n^M$.*

Proof A direct analysis shows that the statement holds for $n \in \{3, 4\}$. Then, we assume $n \geq 5$ for the rest of the proof. If $\dot{G}$ is complete, the statement comes from Proposition 4.6. For otherwise, by Lemma 4.5, there exists a complete unbalanced graph $\dot{K}$ such that $\lambda_1(\dot{G}) \leq \lambda_1(\dot{K})$. We now show that if $\dot{G} \in \mathcal{U}_n$ is not complete, then its index is strictly less than $M(\mathcal{I}_n)$.

Suppose first that $\dot{K} \not\backsimeq \dot{K}_n^M$. In this case, again by Proposition 4.6, we have $\lambda_1(\dot{G}) \leq \lambda_1(\dot{K}) < M(\mathcal{I}_n)$. Therefore, $\dot{G}$ is not a maximizer (for the index) in $\mathcal{U}_n$.

Suppose now that $\dot{K} \backsimeq \dot{K}_n^M$. Along the proof of Lemma 4.5, we have seen that $\dot{K}$ admits an unbalanced subgraph $\dot{H}$ of size $\binom{n}{2} - 1$ such that

$$\lambda_1(\dot{G}) \leq \lambda_1(\dot{H}) \leq \lambda_1(\dot{K}). \tag{10}$$

Our final argument consists in showing that the second inequality of (10) is necessarily strict. In our hypothesis, there exists a permutation (0, 1)-matrix P and a diagonal matrix D such that $P^{-1}D^{-1}A_{\dot{K}}DP = A_{\dot{K}_n^M}$. It is immediately seen that $P^{-1}D^{-1}A_{\dot{H}}DP$ is the adjacency matrix of a graph of type $\dot{H}' = \dot{K}_n^M - e$, which is consequently switching isomorphic to $\dot{H}$.

Note that $\sigma^M(e) = 1$, since for otherwise $\dot{H}'$ (and the switching equivalent $\dot{H}$) would be balanced.

Let $\mathbf{x}$ be a $\lambda_1(\dot{H}')$-eigenvector with positive coordinates (such an eigenvector exists by Lemma 4.4). It is straightforward to check that

$$\lambda_1(\dot{H}') = \mathbf{x}^\intercal A_{\dot{H}'}\mathbf{x} < \mathbf{x}^\intercal A_{\dot{K}_n^M}\mathbf{x},$$

and the latter is not larger than $M(\mathcal{I}_n)$. Hence,

$$\lambda_1(\dot{G}) \leq \lambda_1(\dot{H}) = \lambda_1(\dot{H}') < \lambda_1(\dot{K}_n^M) = \lambda_1(\dot{K}) = M(\mathcal{I}_n),$$

as claimed. □

The last theorem of this section concerns the set of signed graphs minimizing the index in $\mathcal{U}_n$ and completes the results announced in the opening section. As we shall see few lines below, the set (4) comes up once again.

Theorem 4.8 *For $n \geq 3$, let $m(\mathcal{I}_n)$ denote the minimum of the set $\mathcal{I}_n = \{\lambda_1(\dot{G}) \mid \dot{G} \in \mathcal{U}_n\}$. Then $m(\mathcal{I}_n) = 1$ and*

$$\{\dot{G} \in \mathcal{U}_n \mid \lambda_1(\dot{G}) = 1\} = \{\dot{G} \in \mathcal{U}_n \mid \dot{G} \backsimeq -K_n\}.$$

Proof An unbalanced signed graph $\dot{G} = (G, \sigma)$ is surely non-empty, and thus it contains a single edge as an induced subgraph which (by eigenvalue interlacing) yields $\lambda_1(\dot{G}) \geq 1$. Since $\lambda_1(-K_n) = 1$, we infer that $m(\mathcal{I}_n) = 1$.

Assume by way of contradiction that $\lambda_1(\dot{H}) = 1$, for some $\dot{H} \not\backsimeq -K_n$. If $\dot{H}$ is non-complete, then it necessarily contains a 3-vertex path as an induced subgraph, which implies $\lambda_1(\dot{H}) \geq \sqrt{2} > 1$, a contradiction. If $\dot{H}$ is complete then it contains a balanced triangle (this can be seen directly or by consulting (Zaslavsky 1982,Proposition 3.2)) as an induced subgraph giving $\lambda_1(\dot{H}) \geq 2 > 1$, a final contradiction which completes the proof. □

5 Concluding remarks

In this paper we have computed the minimum and the maximum of the sets (1) involving spectral radii and indices respectively. The values of $m(\mathcal{SR}_n)$ can be seen in Table 1, whereas $M(\mathcal{SR}_n) = n - 1$. On the other hand,

$$m(\mathcal{I}_n) = \min \mathcal{I}_n = 1 \quad \text{and} \quad M(\mathcal{I}_n) = \max \mathcal{I}_n = \frac{n + \sqrt{(n-2)(n+6)}}{2} - 2.$$

The reader can easily check that the difference $m(\mathcal{SR}_n) - m(\mathcal{I}_n)$ for $n \geq 8$ strictly increases with n and tends to 1, whereas $M(\mathcal{SR}_n) - M(\mathcal{I}_n)$ strictly decreases and tends to 0.

Unbalanced graphs achieving such extremal values have been all identified and the corresponding results are summarized in Theorems 3.1, 3.5, 4.7 and 4.8. One might ask which are the minimizers and the maximizers for the spectral radius or the index in the set $\mathcal{G}_n$ of all connected signed graphs. The answer is readily available, as extremal graphs in $\mathcal{B}_n = \mathcal{G}_n \setminus U_n$ are known since the dawn of the Spectral Graph Theory. It is, in fact, well-known that the minimizer and the maximizer for both the spectral radius and the index are, up to switching isomorphism, the n-vertex path P_n and the complete graph K_n, respectively. Summing up the conclusions, we list below a representative of each switching isomorphic extremal class in $\mathcal{G}_n$ for $n \geq 1$:

- $-K_n$ minimizes the index;
- K_n maximizes the index;
- both K_n and $-K_n$ maximize the spectral radius.

The description of minimizers of the spectral radius is more subtle. In fact, we find:

- P_3 for $1 \leq n \leq 3$;
- the unbalanced quadrangle C_4^- for $n = 4$;
- $\dot{Q}_{0,1}$ and P_5 for $n = 5$;
- the unbalanced minimizers listed in Theorem 3.5 for $6 \leq n \leq 8$;
- $\dot{Q}_{\lceil \frac{n-4}{2} \rceil, \lfloor \frac{n-4}{2} \rfloor}$ and P_n for odd $n \geq 9$;
- P_n for even $n \geq 10$.

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