Bull. Math. Soc. Sci. Math. Roumanie
Tome 65 (113), No. 3, 2022, 303–315

Notes on Johnson and Hamming signed graphs

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Abstract

We consider relations between symmetric association schemes and strongly regular signed graphs. Our results include constructions of new examples of such signed graphs, relations between their structure and spectrum, and their classification into the known classes. We also propose definitions of Johnson signed graphs and Hamming signed graphs, compute their eigenvalues, and provide necessary and sufficient conditions for their strong regularity. Some constructions of strongly regular Johnson signed graphs with five eigenvalues are provided – according to our knowledge, these are the first examples of strongly regular signed graphs with more than four eigenvalues.

Key Words: Strongly regular signed graph, symmetric association scheme, Johnson graph, Hamming graph, spectrum.
2010 Mathematics Subject Classification: Primary 05C50; Secondary 05C22.

1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{1, -1\}$ is the *sign function* or the *signature*. The number of vertices of $\dot{G}$ is denoted by n. The edge set of $\dot{G}$ is composed of subsets of positive and negative edges which induce the subgraphs denoted by $\dot{G}^+$ and $\dot{G}^-$, respectively. Throughout the paper we interpret a graph as a signed graph with all the edges being positive.

We say that a signed graph is complete, totally disconnected, regular or bipartite if the same holds for its underlying graph. A signed graph is said to be *homogeneous* if all its edges have the same sign or its edge set is empty. Otherwise, it is *inhomogeneous*. The *negation* $-\dot{G}$ of $\dot{G}$ is obtained by reversing the sign of all edges of $\dot{G}$.

A concept of strongly regular signed graphs (for short, SRSGs) has been recently introduced in [10] where the disconnected SRSGs are determined and the bipartite ones are characterized by means of their eigenvalues. The definition of a SRSG is given in the next section. Here we mention that a homogeneous SRSG is an unsigned strongly regular graph or its negation, so in the framework of signed graphs our attention is primarily focused on inhomogeneous ones. In the recent article [6], all inhomogeneous SRSGs are partitioned into the five classes and one class is studied. This class, denoted by $\mathcal{C}_3$, receives an attention in this study, as well. It is noted in [6, 10] that, in contrast to the unsigned case, a SRSG may have more than three (distinct) eigenvalues, and certain examples with four eigenvalues are constructed. According to our knowledge, examples of SRSGs with more than four eigenvalues cannot be found in literature. Sporadic results on SRSGs with two or three eigenvalues can be found in [2, 5, 7, 11]. We mention in passing that every inhomogeneous signed graph with two eigenvalues is strongly regular, and if such a signed graph is complete then its adjacency matrix is the Seidel matrix of an unsigned graph induced by negative edges. The equivalence class of this matrix is the regular two graph equivalence class studied in [9] and also in many other references not listed here. Relations between SRSGs and symmetric 3-class association schemes are investigated in [8].

In this paper we consider SRSGs constructed from symmetric d-class association schemes; in particular, we put our focus on SRSGs that arise for $d = 3$ or $d = 4$. We construct some examples and consider their structural and spectral properties. We also propose definitions of Johnson signed graphs and Hamming signed graphs that arise from Johnson and Hamming schemes, respectively. They act as 'signed' counterparts to the well-known Johnson and Hamming graphs. We compute their eigenvalues, give some examples and establish a necessary and sufficient condition for their strong regularity. We also point out the constructions of two Johnson signed graphs that are strongly regular and have five eigenvalues each.

The paper is organized as follows. Section 2 contains some necessary terminology, notation and conventional concepts. In Section 3 we compute the spectrum of signed graphs that naturally arise from symmetric 3-class association schemes of Johnson, Hamming or rectangular type. In Section 4 we define Johnson and Hamming signed graphs, compute their spectra, determine whether they are strongly regular and give some examples.

2 Preliminaries

For notation not given here and for some basic results on spectra of signed graphs the reader is referred to Zaslavsky's paper [12].

If the vertices i and j of a signed graph are adjacent, we write $i \sim j$; in particular, the existence of a positive (negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ ($i \overset{-}{\sim} j$). The *net-degree* of a vertex i is the difference of the numbers of positive and negative edges incident with it. A signed graph is called *net-regular* if the net-degree considered as a function on the vertex set is a constant.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. The *eigenvalues* of $\dot{G}$ are the eigenvalues of $A_{\dot{G}}$. By [12], the all-1 vector $\mathbf{j}$ is an eigenvector of $\dot{G}$ if and only if $\dot{G}$ is net-regular, and then $\mathbf{j}$ belongs to the eigenspace of its net-degree eigenvalue.

We say that a signed graph $\dot{G}$ is *strongly regular* (for short, $\dot{G}$ is a *SRSG*) whenever it is neither homogeneous complete nor totally disconnected and there exist $r \in \mathbb{N}$, $a, b, c \in \mathbb{Z}$, such that the entries of $A_{\dot{G}}^2$ satisfy

$$a_{ij}^{(2)} = \begin{cases} r & \text{if } \ i = j, \\ a & \text{if } \ i \overset{+}{\sim} j, \\ b & \text{if } \ i \overset{-}{\sim} j, \\ c & \text{if } \ i \nsim j \text{ and } i \neq j. \end{cases}$$

Obviously, $\dot{G}$ is regular with vertex degree r. Observe that $a_{ij}^{(2)}$ is the difference of the numbers of positive and negative i-j walks of length 2 in $\dot{G}$. Accordingly, this definition generalizes the definition of a strongly regular graph. It can be expressed in the matrix form as

$$A_{\dot{G}}^2 = \frac{a}{2}(A_{\dot{G}} + A_G) - \frac{b}{2}(A_{\dot{G}} - A_G) + cA_{\overline{G}} + rI,$$

where $\overline{G}$ is the complement of G.

We already mentioned that our attention is focused on connected inhomogeneous SRSGs. If $\dot{G}$ is inhomogeneous, the parameters a, b are uniquely determined. If, in addition, $\dot{G}$ is non-complete then c is fixed, as well, and in this case we usually write the parameters of $\dot{G}$ in the form of the ordered quintuple (n, r, a, b, c).

We proceed with association schemes. Let d be a positive integer and X a non-empty set. A symmetric *d-class association scheme* on X consists of a partition of $X \times X$ into $d+1$ non-empty symmetric binary relations $R_0, R_1, \dots, R_d$ satisfying the following conditions:

- $R_0 = \{(x, x) \mid x \in X\}$,
- if $(x, y) \in R_h$, then the number $z \in X$ such that $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant p_{ij}^h depending on i, j, h, but not on a choice of x, y.

The numbers p_{ij}^h are called the *intersection numbers* of the scheme. The ith binary relation R_i, also known as the ith associate class, can be represented by the matrix A_i in the following way: A_i is the $(0, 1)$-matrix of order $|X|$ whose rows and columns are indexed by the elements of X, and $(A_i)_{xy} = 1$ if and only if $(x, y) \in R_i$. It follows that $A_0 = I$, $A_i = A_i^\intercal$, $\sum_{h=0}^{d} A_h = J$ (the all-1 matrix), $A_iA_j = \sum_{k=0}^{d} p_{ij}^h A_h$. These matrices are also known as the *adjacency matrices* of the association scheme.

To make the paper more self-contained, we give more details on the aforementioned matrices and simultaneously introduce some notation. These details can be found in [3], and an experienced reader might skip this part and take into account only the last sentence of this paragraph. It follows that the real span $\langle \mathcal{A} \rangle$ of $A_0, A_1, \dots, A_d$ is a $(d+1)$-dimensional commutative algebra of symmetric matrices, the so-called *Bose-Mesner algebra* of the association scheme. Since these matrices are symmetric and commute pairwise, they can be diagonalized simultaneously and the entire space $\mathbb{R}^{|X|}$ can be expressed as a direct sum of their common eigenspaces. Since the dimension of $\langle \mathcal{A} \rangle$ is $d+1$, there are exactly $d+1$ (common) eigenspaces of dimension f_j, $0 \leq j \leq d$. Note that one of the eigenspaces has dimension 1 (as the all-1 matrix J belongs to $\langle \mathcal{A} \rangle$ and $|X|$ is its eigenvalue of multiplicity 1), so without loss of generality we may assume that $f_0 = 1$. Denote by $E_0, E_1, \dots, E_d$ the orthogonal projections of $\mathbb{R}^{|X|}$ onto those $d+1$ eigenspaces. We have $E_0 = |X|^{-1}J$ and $E_iE_j = \delta_{ij}E_i$. The sets $\{A_0, A_1, \dots, A_d\}$ and $\{E_0, E_1, \dots, E_d\}$ are bases for $\langle \mathcal{A} \rangle$ (cf. [3, Theorem 2.6.1]), so there exist the numbers $p_i(j)$ such that $A_i = \sum_j p_i(j)E_j$, and there exist the numbers $q_j(i)$ such that $E_j = |X|^{-1} \sum_i q_j(i)A_i$. The numbers $p_i(j)$ are known as the *eigenvalues* of the association scheme. If P and Q are the $(d+1) \times (d+1)$ matrices defined by $(P)_{i,j} = p_j(i)$ and $(Q)_{i,j} = q_j(i)$, then $PQ = QP = |X|I$. It also holds $A_iE_j = p_i(j)E_j$, so $p_i(j)$ is an eigenvalue of A_i with multiplicity $f_j = \mathrm{rank}(E_j)$.

The *d-class Johnson scheme* $J(v, d)$ is defined on the d-subsets of a v-set, with $d \leq \frac{v}{2}$. Two d-subsets are in relation R_i if and only if they intersect in $d - i$ elements. The eigenvalues of the Johnson scheme are

$$p_i(j) = \sum_{r=0}^{i} (-1)^{i-r} \binom{d-r}{i-r} \binom{v-d+r-j}{r} \binom{d-j}{r} \tag{2.1}$$

and their multiplicities are $m_i = \binom{v}{i} - \binom{v}{i-1}$, along with the convention that $\binom{v}{-1} = 0$ [4].

The *d-class Hamming scheme* $H(d,q)$ has the set of all words of length d over an alphabet of q symbols as its vertex set. Two words are in relation R_i if and only if the Hamming distance between them is i (i.e., if and only if they differ in exactly i coordinates). The eigenvalues of the Hamming scheme are

$$p_i(j) = \sum_{r=0}^{i} (-1)^r (q-1)^{i-r} \binom{d-j}{i-r} \binom{j}{r} \tag{2.2}$$

and their multiplicities are $m_i = (q-1)^i \binom{d}{i}$ [4].

If m and n are positive integers, then the (3-class) *rectangular scheme* $R(m,n)$ has as vertices the ordered pairs (i,j), where $1 \leq i \leq m,\ 1 \leq j \leq n$. For two ordered pairs one of the following may occur: They agree in the first coordinate, in which case they are first associates (that is, they are in relation R_1), they agree in the second coordinate, in which case they are second associates, or they do not agree in any coordinate, in which case they are third associates.

3 Constructing SRSGs from particular 3-class association schemes

Here we start from a signed graph and arrive at related symmetric binary relations that are similar to those defined in the previous section.

Obviously, every signed graph $\dot{G}$ defines the four symmetric binary relations on its vertex set V:

$$\begin{aligned}
R_0 &= \{(x,y) \in V \times V \mid x = y\},\\
R_1 &= \{(x,y) \in V \times V \mid x \overset{+}{\sim} y\},\\
R_2 &= \{(x,y) \in V \times V \mid x \overset{-}{\sim} y\},\\
R_3 &= \{(x,y) \in V \times V \mid x \nsim y,\ x \neq y\}.
\end{aligned}$$

We have $V \times V = R_0 \cup R_1 \cup R_2 \cup R_3$. Observe that any of the relations R_1, R_2, R_3 can be empty; for example, if just R_2 is empty, then $\dot{G}$ is homogenous with all edges being positive.

For two vertices $x, y \in V$ satisfying $(x,y) \in R_h$, with $h \in \{0,1,2,3\}$, we define the parameters:

$$\begin{aligned}
p_{11}^h(x,y) &= |\{z \in V \mid (x,z) \in R_1, (y,z) \in R_1\}|,\\
p_{22}^h(x,y) &= |\{z \in V \mid (x,z) \in R_2, (y,z) \in R_2\}|,\\
p_{12}^h(x,y) &= |\{z \in V \mid (x,z) \in R_1, (y,z) \in R_2\}|,\\
p_{21}^h(x,y) &= |\{z \in V \mid (x,z) \in R_2, (y,z) \in R_1\}|.
\end{aligned}$$

Note that $p_{12}^0(x,y) = p_{21}^0(x,y) = 0$ holds. Now, $\dot{G}$ is strongly regular if at most one of the relations R_1, R_2, R_3 is empty and if for all $x, y \in V$ and all $h \in \{0,1,2,3\}$ such that $(x,y) \in R_h$, the parameter

$$p_{11}^h(x,y) + p_{22}^h(x,y) - p_{12}^h(x,y) - p_{21}^h(x,y)$$

depends on h, but not on x or y. Clearly, the conditions given in the definition of a symmetric 3-class association scheme are stronger than the ones above. Therefore, from a given symmetric 3-class association scheme, we can easily construct a SRSG in the following way: Any of the relations R_1, R_2, R_3 of the scheme can be chosen to represent positive edges, any of the remaining two can be chosen to represent negative ones, and then the remaining relation represents non-edges. Indeed, the signed graph constructed in this way is strongly regular, because the sets R_1, R_2, R_3 are disjunct and non-empty, and the intersection numbers p_{ij}^h depend only on i, j, h. If we choose R_i to represent positive edges and R_j to represent negative ones, and if A_i and A_j are the the corresponding matrices, then the adjacency matrix of $\dot{G}$ is $A_{\dot{G}} = A_i - A_j$, while the parameters a, b, c of $\dot{G}$ can be expressed in terms of the intersection numbers in the following way:

$$a = p_{11}^i + p_{22}^i - 2p_{12}^i,\ b = p_{11}^j + p_{22}^j - 2p_{12}^j,\ c = p_{11}^k + p_{22}^k - 2p_{12}^k, \tag{3.1}$$

where we use that in a symmetric association scheme $p_{ij}^h = p_{ji}^h$ holds true [3, Lemma 2.1.1]. Such constructions can also be found in [8]. Note that a SRSG constructed in this way is net-regular with net-degree $p_{ii}^0 - p_{jj}^0$.

In [8] one can find explicit values of the parameters of SRSGs constructed on the basis of a 3-class Johnson scheme $J(v,3)$, or a 3-class Hamming scheme $H(3,q)$, or a rectangular scheme $R(m,n)$. We extend these results by computing the eigenvalues of the corresponding SRSGs and present them in Tables 1-3. We use $\dot{G}_{i,j}$ to denote the SRSG with the adjacency matrix $A_{\dot{G}} = A_i - A_j$, and take into account only SRSGs with $i < j$, since $\dot{G}_{i,j}$ is $-\dot{G}_{j,i}$ and, according to [6, Lemma 2.1], the negation of a SRSG is again a SRSG with the parameters a and b interchanged. We remark that the parameter b in this paper and the one defined in [8] differ in sign.

Observe that a SRSGs obtained as in Tables 1-3 have at most four eigenvalues. Thus, we may inspect whether they have three or less eigenvalues.

Equating expressions in every column of Table 1, we get that there are four SRSGs $\dot{G}_{1,2}$ with three eigenvalues. They are obtained from $J(6,3)$, $J(7,3)$, $J(9,3)$ and $J(14,3)$. SRSGs $\dot{G}_{1,3}$ obviously have three eigenvalues, except when they are obtained from $J(7,3)$ or $J(9,3)$. In these two cases we get SRSGs with two eigenvalues (8 with multiplicity 7 and -2 with multiplicity 28, and 19 with multiplicity 8 and -2 with multiplicity 76, respectively). There are also three SRSGs $\dot{G}_{2,3}$ with three eigenvalues. They are obtained from $J(6,3)$, $J(8,3)$ and $J(11,3)$.

Table 1: Eigenvalues of $\dot{G}_{i,j}$ that arises from the Johnson scheme $J(v,3)$, for $v \geq 6$.

Eigenvalue	$\dot{G}_{1,2}$	$\dot{G}_{1,3}$	$\dot{G}_{2,3}$	Multiplicity
λ_0	$-\frac{3(v-3)(v-6)}{2}$	$-\frac{(v-3)(v^2-9v+2)}{6}$	$-\frac{(v-3)(v-4)(v-14)}{6}$	1
λ_1	$-\frac{v^2-17v+54}{2}$	$\frac{(v-2)(v-3)}{2} - 2$	$(v-4)(v-7)$	$v-1$
λ_2	$3v-18$	-2	$16-3v$	$\frac{v(v-3)}{2}$
λ_3	-6	-2	4	$\frac{v(v-1)(v-5)}{6}$

Considering Table 2, we conclude that the corresponding SRSGs $\dot{G}_{1,2}$ have four eigenvalues, except in cases in which they are obtained from $H(3,2)$ or $H(3,6)$ (when they have

Table 2: Eigenvalues of $\dot{G}_{i,j}$ that arises from the Hamming scheme $H(3,q)$.

Eigenvalue	$\dot{G}_{1,2}$	$\dot{G}_{1,3}$	$\dot{G}_{2,3}$	Multiplicity
λ_0	$-3(q-1)(q-2)$	$3q-(q-1)^3-3$	$-(q-1)^2(q-4)$	1
λ_1	$-q^2+6q-6$	q^2-2	$2(q-1)(q-2)$	$3(q-1)$
λ_2	$3q-6$	-2	$4-3q$	$3(q-1)^2$
λ_3	-6	-2	4	$(q-1)^3$

Table 3: Eigenvalues of $\dot{G}_{i,j}$ that arises from the rectangular scheme $R(m,n)$.

Eigenvalue	$\dot{G}_{1,2}$	$\dot{G}_{1,3}$	$\dot{G}_{2,3}$	Multiplicity
λ_0	$n-m$	$(n-1)(2-m)$	$(m-1)(2-n)$	1
λ_1	0	-2	-2	$(m-1)(n-1)$
λ_2	$-m$	$m-2$	$2m-2$	$n-1$
λ_3	n	$2n-2$	$n-2$	$m-1$

three eigenvalues) or $H(3,3)$ (when they have two eigenvalues: 3 with multiplicity 18 and -6 with multiplicity 9). SRSGs $\dot{G}_{1,3}$ have three eigenvalues, except when they are obtained from $H(3,2)$ or $H(3,3)$ (and there they have two eigenvalues: 2 and -2 both with multiplicity 4 for $H(3,2)$, and 7 with multiplicity 6 and -2 with multiplicity 21 for $H(3,3)$). Every SRSG $\dot{G}_{2,3}$ has four eigenvalues, unless is obtained from $H(3,3)$ when it has only two eigenvalues, 4 with multiplicity 15 and -5 with multiplicity 12.

For Table 3, SRSGs $\dot{G}_{1,2}$ have four eigenvalues, except for $n=m$ when they have three eigenvalues. SRSGs $\dot{G}_{1,3}$ have three eigenvalues if they are obtained from $R(3,3)$ or $R(2n,n)$ with $n \neq 2$. Two eigenvalues occur only for the SRSG obtained from $R(4,2)$ (2 and -2, both with multiplicity 4). Similarly, SRSGs $\dot{G}_{2,3}$ have three eigenvalues if they are obtained from $R(3,3)$ or $R(m,2m)$ with $m \neq 2$. Two eigenvalues occur for $R(2,4)$ (again, 2 and -2, both with multiplicity 4).

We have mentioned in the first section that in [6] all inhomogeneous SRSGs are divided into five classes. We conclude this section by considering a particular class denoted in [6] by $\mathcal{C}_3$. We know from the same reference that SRSGs belonging to this class have the following properties: they have three eigenvalues, they are net-regular and their net-degree considered as an eigenvalue has multiplicity 1. Hence, from the spectral perspective they resemble unsigned strongly regular graphs. By definition, a SRSG belongs to $\mathcal{C}_3$ if its parameters satisfy $a+b \neq 0$ and, if a signed graph is non-complete, $c=\frac{a+b}{2}$. Complete SRSGs of $\mathcal{C}_3$ have received a great deal of attention in the mentioned reference where one can find some theoretical results and many constructions of such SRSGs. An interesting problem is the construction of non-complete SRSGs of the same class. Two constructions are given [6]. These are the SRSGs $\dot{G}_{1,2}$ obtained from 3-class Johnson schemes $J(9,3)$ and $J(14,3)$, respectively. Considering Tables 1-3 and taking into account the parameters condition which can be computed by (3.1), we easily get more examples. Those obtained from 3-class Johnson schemes (and not mentioned in [6]) are the infinite family of SRSGs $\dot{G}_{1,3}$ obtained from $J(v,3)$ with $v=8$ or $v \geq 10$, and two SRSGs $\dot{G}_{2,3}$ obtained from $J(6,3)$

and $J(8,3)$. Similarly, 3-class Hamming schemes give $\dot{G}_{1,2}$ obtained from $H(3,6)$ and the infinite family $\dot{G}_{1,3}$ obtained from $H(3,q)$ with $q \notin \{2,3\}$. The rectangular schemes give the infinite family of SRSGs $\dot{G}_{1,3}$ obtained from $R(2n,n)$ with $n \neq 2$, and one other family $\dot{G}_{2,3}$ obtained from $R(m,2m)$ with $m \neq 2$.

4 Johnson signed graphs and Hamming signed graphs

Inspired by the previous section, we propose the definitions of Johnson signed graphs and Hamming signed graphs.

Let $J(v,d)$, $2 \leq d \leq \frac{v}{2}$, be the Johnson scheme and let $A_1, A_2, \ldots, A_d$ be its adjacency matrices. The *Johnson signed graph* $J(v,d)_{m,n}$, for $1 \leq m,n \leq d$ and $m \neq n$, is the signed graph determined by the adjacency matrix $A_{J(v,d)_{m,n}} = A_m - A_n$.

Similarly, let $H(d,q)$, $d,q \geq 2$, be the Hamming scheme and let $A_1, A_2, \ldots, A_d$ be its adjacency matrices. The *Hamming signed graph* $H(d,q)_{m,n}$, for $1 \leq m,n \leq d$ and $m \neq n$, is the signed graph determined by the adjacency matrix $A_{H(d,q)_{m,n}} = A_m - A_n$.

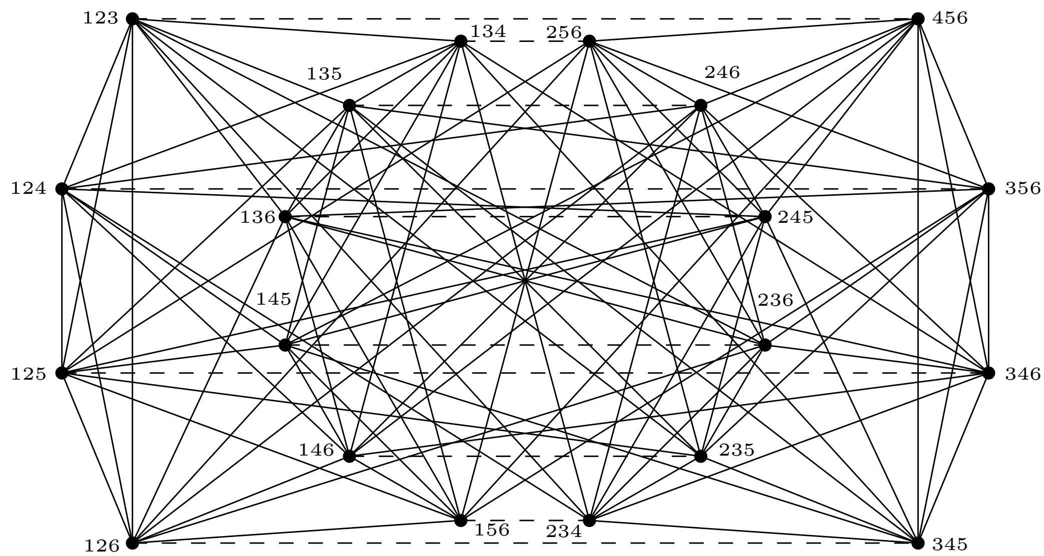
Note that for a signed graph $\dot{G}$ defined above the adjacency matrix of $\dot{G}^+$ ($\dot{G}^-$) is A_m (A_n) and, since for $m \neq n$ the matrices A_m and A_n are mutually disjoint, $\dot{G}$ is never totally disconnected. None of the adjacency matrices of the association scheme is a zero matrix, so $\dot{G}$ is always inhomogeneous. Observe also that by interchanging m and n we get $-\dot{G}$. In both cases $\dot{G}$ is net-regular with net-degree $p^0_{mm} - p^0_{nn}$ and its underlying graph is regular with vertex degree $p^0_{mm} + p^0_{nn}$.

For a fixed d, there are exactly $\binom{d}{2}$ Johnson (Hamming) signed graphs defined as above. Thus, if $d = 2$ we can define only one Johnson signed graph $J(v,2)_{1,2}$ and one Hamming signed graph $H(2,q)_{1,2}$: the adjacency matrix of $J(v,2)_{1,2}$ is the Seidel matrix of the complement of the triangular graph $T(v)$ (i.e., the line graph of K_v), and the adjacency matrix of $H(2,q)_{1,2}$ is the Seidel matrix of the complement of the $q \times q$ grid graph (i.e., the line graph of $K_{q,q}$). For $d \geq 3$, we get non-complete Johnson and Hamming signed graphs. Those with the smallest number of vertices are obtained for $(v,d) = (6,3)$ and $(d,q) = (3,2)$, and they have 20 and 8 vertices, respectively. Examples are illustrated in Figure 1.

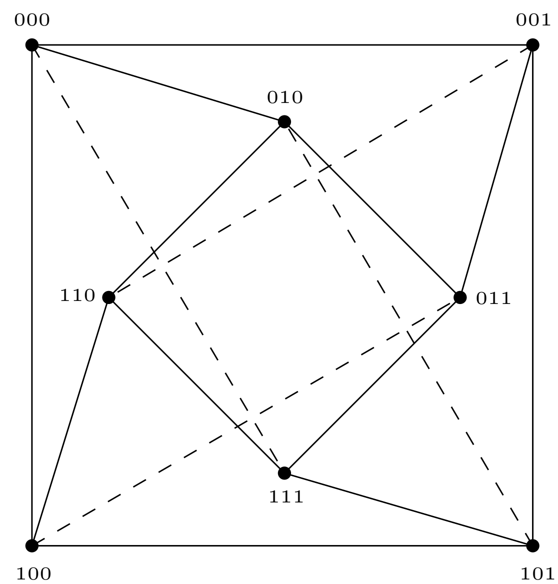
Remark 1. Recall from Section 2 that the vertices of an unsigned Johnson graph are identified with the d-subsets of a v-set, and two vertices are adjacent if and only if the size of the intersection of the corresponding subsets is $d-1$. In other words, the adjacency matrix of a Johnson graph is the matrix A_1 of the d-class Johnson scheme. However, there is a generalization of a Johnson graph in which two vertices are adjacent if and only if the intersection size is equal to i, where i is a fixed integer from 1 to $d-1$ [1]. Now, many definitions in the framework of signed graphs encapsulate their 'unsigned' variants. For example, SRSGs include strongly regular graphs (and their negations). To do the same in the case of Johnson signed graphs, one should include the possibility that exactly one of the matrices A_m, A_n is a zero matrix or say that a Johnson signed graph is either a generalized Johnson graph, its negation or its adjacency matrix is $A_m - A_n$.

With slight modifications, all the previous remarks apply to Hamming signed graphs.

We proceed with some theoretical results. In the first one we compute the eigenvalues of the defined signed graphs.



$J(6,3)_{1,3}$



$H(3,2)_{1,3}$

Figure 1: The Johnson signed graph $J(6,3)_{1,3}$ and the Hamming signed graph $H(3,2)_{1,3}$. Negative edges are dashed.

Proposition 1. *The following statements hold true.*

(i) The eigenvalues λ_j, $0 \leq j \leq d$, of the Johnson signed graph $J(v,d)_{m,n}$ are given by:

$$\lambda_j = \sum_{r=0}^{m}(-1)^{m-r}\binom{d-r}{m-r}\binom{v-d+r-j}{r}\binom{d-j}{r} -$$

$$-\sum_{r=0}^{n}(-1)^{n-r}\binom{d-r}{n-r}\binom{v-d+r-j}{r}\binom{d-j}{r};$$

(ii) The eigenvalues λ_j, $0 \leq j \leq d$, of the Hamming signed graph $H(d,q)_{m,n}$ are given by:

$$\lambda_j = \sum_{r=0}^{m}(-1)^{r}(q-1)^{m-r}\binom{d-j}{m-r}\binom{j}{r} - \sum_{r=0}^{n}(-1)^{r}(q-1)^{n-r}\binom{d-j}{n-r}\binom{j}{r}.$$

Proof. Both results follow from definitions of the corresponding signed graphs, the facts that the matrices of an association scheme can be simultaneously diagonalized and the formulas (2.1) and (2.2). ∎

We now consider the question of strong regularity.

Proposition 2. *A Johnson (Hamming) signed graph $J(v,d)_{m,n}$ ($H(d,q)_{m,n}$) is strongly regular if and only if $p^h_{mm} + p^h_{nn} - 2p^h_{mn}$ is a constant not depending on h, for $1 \leq h \leq d$ and $h \notin \{m,n\}$.*

Proof. Denote the signed graph under consideration by $\dot{G}$, and let i, j be its vertices.

We record the following observations. First, $i \overset{+}{\sim} j$ holds precisely when i and j are in relation m. It follows that the difference of the numbers of positive and negative i-j walks of length 2 is $p^m_{mm} + p^m_{nn} - p^m_{mn} - p^m_{nm} = p^m_{mm} + p^m_{nn} - 2p^m_{mn}$. By definition of a symmetric association scheme, this is a constant, say a, not depending on the choice of i, j. Similarly, $i \overset{-}{\sim} j$ holds precisely when i and j are in relation n and the mentioned difference is now equal to $p^n_{mm} + p^n_{nn} - 2p^n_{mn}$. Again this is a constant, say b, not depending on the choice of i, j.

Assume now that $\dot{G}$ is strongly regular. According to the previous observations, it remains to consider the case $i \nsim j$. Clearly, $i \nsim j$ holds precisely if i, j are in a relation $h \notin \{m,n\}$. The difference of the numbers of positive and negative i-j walks of length 2 is $p^h_{mm} + p^h_{nn} - 2p^h_{mn}$. Since $\dot{G}$ is strongly regular, the last expression has the same value for every $h \notin \{m,n\}$, and we are done.

Conversely, if $p^h_{mm} + p^h_{nn} - 2p^h_{mn}$ is a constant not depending on $h \notin \{m,n\}$, then the mentioned difference is also a constant for $i \nsim j$, which together with the initial observations implies the strong regularity of $\dot{G}$. ∎

The parameters of a strongly regular Johnson signed graph are

$$\left(\binom{v}{d}, p^0_{mm} + p^0_{nn}, p^m_{mm} + p^m_{nn} - 2p^m_{mn}, p^n_{mm} + p^n_{nn} - 2p^n_{mn}, p^h_{mm} + p^h_{nn} - 2p^h_{mn}\right),$$

where $1 \leq h \leq d$ and $h \notin \{m, n\}$. Similarly, for a strongly regular Hamming signed graph, we have

$$\left(q^d,\, p^0_{mm} + p^0_{nn},\, p^m_{mm} + p^m_{nn} - 2p^m_{mn},\, p^n_{mm} + p^n_{nn} - 2p^n_{mn},\, p^h_{mm} + p^h_{nn} - 2p^h_{mn}\right),$$

where $1 \leq h \leq d$ and $h \notin \{m, n\}$. Both sets of parameters are computed on the basis of the proof of the previous result.

The previous proposition gives a criterion for strong regularity which is elegant, but based on an expression whose computation can be tedious in some situations. Therefore, we continue with some sufficient conditions for non-strong regularity. Recall that both Johnson and Hamming schemes are metric (with respect to the ordering of their binary relations), which means that we have $p^h_{ij} = 0$ unless $|i - j| \leq h \leq i + j$, and $p^h_{ij} > 0$ whenever $i + j = h \leq k$ $(i + j = h \leq d)$; see [4, p. 327]. The next result is a consequence of these facts.

Proposition 3. *If $m < n < \frac{d}{2}$, then a Johnson (Hamming) signed graph $J(v, d)_{m,n}$ $(H(d, q)_{m,n})$ is not strongly regular.*

Proof. Suppose first that $d = 2t$. Then $m < n \leq t - 1$, so for $z = 2n + 1$ and $s = 2n$ we have $z, s \leq d$ and $z, s \notin \{m, n\}$. We also have $p^z_{mm} + p^z_{nn} - 2p^z_{mn} = 0$. Namely, every term on the left hand side is zero, since $2m < m + n < 2n < 2n + 1 = z$ and the scheme under consideration is metric. On the other hand, we have $p^s_{mm} + p^s_{nn} - 2p^s_{mn} = p^{2n}_{nn} > 0$, which means that $p^h_{mm} + p^h_{nn} - 2p^h_{mn}$ is not a constant, and so $\dot{G}$ is not strongly regular by Proposition 2.

Similarly, if d is odd, then $d = 2t + 1$ and $m < n \leq t$. Again, we take $z = 2n + 1$, $s = 2n$, when we obviously have the same constraints for z, s as before. Here we have $2m < m+n < 2n < 2n+1 = z$, thus $p^z_{mm}+p^z_{nn}-2p^z_{mn} = 0$, but $p^l_{mm}+p^l_{nn}-2p^l_{mn} = p^{2n}_{nn} > 0$, and so $p^h_{mm} + p^h_{nn} - 2p^h_{mn}$ is not a constant, which concludes the proof. ∎

For example, the previous proposition implies that $J(v, d)_{1,2}$ and $H(d, q)_{1,2}$ are not strongly regular for $d > 4$.

In what follows we consider strong regularity of Johnson and Hamming signed graph that arise from the 4-class schemes.

Proposition 4. *For $m < n$, the Johnson signed graph $J(v, 4)_{m,n}$, $v \geq 8$, is strongly regular if and only only if $(v, m, n) \in \{(12, 1, 2), (8, 3, 4)\}$.*

Proof. We need to consider the signed graphs $J(v, 4)_{m,n}$, where $(m, n) \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

Let first $(m, n) = (1, 2)$. By Proposition 2, $J(v, 4)_{1,2}$ is strongly regular if and only if $p^h_{11} + p^h_{22} - 2p^h_{12}$ is a constant not depending on h, for $h \in \{3, 4\}$. We have that $p^3_{11} = p^4_{11} = p^4_{12} = 0$ because $J(v, 4)$ is metric, and so it remains to compute the parameters p^3_{22}, p^3_{12} and p^4_{22}. In $J(v, 4)$, the number p^3_{22} is obtained by fixing two 4-element sets A, B such that $|A \cap B| = 1$, and counting the number of the sets C such that $|A \cap C| = |B \cap C| = 2$. After a short computation, we get $p^3_{22} = 9(v - 7) + 9 = 9(v - 6)$. The only difference for p^3_{12} and p^4_{22} is that in the first case we have $|A \cap C| = 1$ and in the second $|A \cap B| = 0$. In both cases, we get constants (not depending on v): $p^3_{12} = 9$, $p^4_{22} = 36$. Now, $J(v, 4)_{1,2}$ is strongly

Table 4: Parameters and eigenvalues of SRSGs obtained in Propositions 4 and 5.

	$J(12,4)_{1,2}$	$J(8,4)_{3,4}$	$H(4,4)_{1,2}$	$H(4,3)_{2,3}$	$H(4,2)_{2,4}$
n	495	70	256	81	16
r	200	17	66	56	7
a	52	0	2	5	2
b	36	0	10	1	6
c	36	4	6	-6	0
λ_0	18^{54}	15	6^{162}	10^{24}	5^2
λ_1	14^{154}	3^{28}	$(-10)^{93}$	1^{24}	1^8
λ_2	$(-10)^{275}$	1^{20}	-42	$(-8)^{33}$	$(-3)^6$
λ_3	$(-22)^{11}$	$(-5)^{14}$			
λ_4	-136	$(-7)^7$			

regular if and only if $p_{22}^3 - 2p_{12}^3 = p_{22}^4$, so $9(v-6) - 18 = 36$, which means that $v = 12$. Thus, $J(v,4)_{1,2}$ is a SRSG if and only if $v = 12$.

We next consider $(m,n) = (1,3)$. By Proposition 2, $J(v,4)_{1,3}$ is strongly regular if and only if $p_{11}^2 + p_{22}^2 - 2p_{12}^2 = p_{11}^4 + p_{22}^4 - 2p_{12}^4$. In a similar way, we get $4 + 2\binom{v-6}{3} + 4\binom{v-6}{2} - 4(v-6) = 16\binom{v-8}{2} - 32$, but this equation has no integral solutions, which eliminates this possibility.

The remaining four cases are considered in a very similar way, and only $(m,n) = (3,4)$ leads to the solution $(v,m,n) = (8,3,4)$, which completes the proof. □

The next result is obtained in a very similar way.

Proposition 5. *For $m < n$, the Hamming signed graph $H(4,q)_{m,n}$ is strongly regular if and only if $(q,m,n) \in \{(4,1,2),(3,2,3),(2,2,4)\}$.*

In Table 4 we give the data on SRSGs obtained in the previous two propositions. The eigenvalues are ordered decreasingly and their multiplicities are indicated. Observe that $H(4,2)_{2,4}$ is disconnected with two complete components.

Until now, the existence of SRSGs with at most four eigenvalues has been confirmed. In fact, every inhomogeneous signed graph with two eigenvalues is a SRSG with $a = -b$ and, if it is non-complete, $c = 0$ [10, Theorem 4.1]). SRSGs with three or four eigenvalues can be found in [6, 8, 10] and in the previous section. However, according to our knowledge, no examples with five (or more) eigenvalues were known. By analyzing Table 4, we get two examples with five eigenvalues, each being constructed on the basis of 4-class Johnson schemes, and according to the introduced definition, they are Johnson signed graphs.

We continue with the analysis of SRSGs belonging to the class $\mathcal{C}_3$ started in the previous section. Constructions of such signed graphs reported in [6] are based on symmetric 2-class association schemes, and they produce complete SRSGs. In [6, 8] and the previous section, one can find constructions that are based on symmetric 3-class association schemes. We remark that the graph $H(4,4)_{1,2}$ is constructed in an entirely different way. It is a Hamming signed graph obtained by merging R_3 and R_4 of the 4-class Hamming scheme $H(4,4)$, where such a merging does not reduce the scheme to a 3-class one (since, for example, $p_{22}^4 = 6$

and $p^3_{22} = 12$). It would be interesting to know are there more Johnson or Hamming signed graphs of $\mathcal{C}_3$ that are constructed from d-class Johnson or Hamming schemes for $d \geq 5$. Furthermore, can we construct a SRSG $\dot{G}$ of $\mathcal{C}_3$, such that $\dot{G}^+$ and $\dot{G}^-$ are regular with non-commuting adjacency matrices?

Finally, we complete the classification of the remaining SRSGs of Table 4. Definitions of the remaining classes are given [6]. Hence, we have $J(12,4)_{1,2}, H(4,3)_{2,3} \in \mathcal{C}_5$, $J(8,4)_{3,4} \in \mathcal{C}_1$ and $H(4,2)_{2,4} \in \mathcal{C}_4$, while its components belong to $\mathcal{C}_3$.

Acknowledgement. This work is partially supported by the Serbian Ministry of Education, Science and Technological Development via the University of Belgrade.

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Received: 03.06.2021
Revised: 21.09.2021
Accepted: 22.09.2021

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