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On Graphs with Small Ranks: Old and New Results

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Abstract: The *M-rank* of a graph G is the rank of the associated matrix $M(G)$. In this paper, we summarize the results about the graphs with a comparatively small M-rank, and propose some problems for further study. In addition, for a real number $\alpha \in [0, 1]$, we determine all graphs with A_α-rank 2, where $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ and $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of G.

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0 Introduction

For a graph $G = (V(G), E(G))$, let $M(G)$ be a corresponding real or complex square matrix defined in a prescribed way. The *M-spectrum* of G is the multiset $\mathrm{Sp}_M(G)$ of the eigenvalues of $M(G)$. We believe that the reader is familiar with the three standard matrices associated with graphs: the *adjacency matrix* $A(G) = (a_{ij})_{n\times n}$ (with $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$, otherwise), the *Laplacian matrix* $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees.

Different graph matrices play different roles in solving problems from other disciplines. For example, Kirchhoff[22] used the L-matrix to establish the Matrix-Tree Theorem in Electric Networks. Recently, Huang[18] resolved the Sensitivity Conjecture from Computer Science by means of the A-matrix of signed graphs. Jiang et al.[21] employed the second largest eigenvalue of the A-matrix to investigate the equiangular line systems. So far, several new graph matrices have been successively introduced. Nikiforov[31] adopted a unified approach to A-matrix and Q-matrix of G by introducing the A_a-matrix as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, for a real number $\alpha \in [0, 1]$. Clearly, $A_0(G) = A(G)$ and $A_{1/2}(G) = \frac{1}{2}Q(G)$. Meanwhile, another series of graph matrices are based on the well-known distance matrix including distance Laplacian and distance signless Laplacian matrices, see [10, 20] for more details. For further generalizations one may

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consult [9] (generalized adjacency matrix) and [16] (universal adjacency matrix). Apart from that, more graph matrices will be introduced in the next sections.

In this paper, we focus on the rank of graphs. Traditionally, the *M-rank* of G, denoted by $\mathrm{rank}_M(G)$, is the rank of the matrix $M(G)$. In broad literature one can find that A-rank is putted in the first plan, and usually "the rank of a graph" refers to the rank of its adjacency matrix. Similarly, the *nullity* $\eta(G)$ of a graph G is the nullity of its adjacency matrix, i.e., the multiplicity of the zero eigenvalue of $A(G)$. Clearly, $\eta(G) = |V(G)| - \mathrm{rank}_A(G)$ which indicates that the A-rank and $\eta(G)$ are essentially equivalent parameters.

There are several motivations for the study of the rank or the nullity of a graph. For example, by [2, 7], a conjugated hydrocarbon can be represented by its molecular graph G. In the Hückel theory, graphs under consideration are connected and planar with maximum vertex degree 3 (see [38, p. 89]). In this setting, the eigenvectors of $A(G)$ coincide with the Hückel molecular orbitals, and the eigenvalues of $A(G)$ are the energies corresponding to the Hückel molecular orbitals. The number of non-bonding molecular orbitals (NBMOs) is exactly $\eta(G)$. If $\eta(G) > 0$ or equivalently $\mathrm{rank}_A(G) < |V(G)|$, then the molecule corresponding to G has NBMOs in the Hückel spectrum, and such molecule should have open-shell ground states and be very reactive.

In addition, if $\mathrm{rank}_A(G) < |V(G)|$, then G is said to be *singular* (or *non-invertible*); otherwise, G is *nonsingular* (or *invertible*). In 1957, Collatz and Sinogowitz[4] proposed the problem of determining all singular graphs. Obviously, this problem cannot be solved once and for all, and only particular results are known, some of which are reported in [13, 15, 44]. In the other direction, when G is nonsingular, the problem of characterizing the inverse of G was also motivated by the research in the chemical domain (cf. [14]), and it originated from Godsil's study on the least positive eigenvalue of $A(G)$ (cf. [12]). However, we do not define the inverse of a graph, since there are various definitions, see [12, 29, 36, 42–43].

Throughout the paper we write J, O and $\mathbf{0}$ to denote the all-1 matrix, the all-0 matrix, and the all-0 vector, respectively. The size of the matrix may be given in the subscript. As usual, let P_n, C_n, K_n and $K_{n_1,n_2,\cdots,n_k}$ be respectively the *path*, the *cycle*, the *complete graph* and the *complete k-partite graph*. For the graphs G and H, we write kG for the disjoint union of k copies of G, $G \cup H$ for the disjoint union of G and H, and $G \vee H$ for the join of these graphs, i.e., the graph obtained by inserting an edge between every vertex of G and every vertex of H. For the remaining terminology and undefined notions, we refer the reader to [6, 8].

The paper is organized as follows. In Section 1 we give a survey of known results treating graphs (or their generalizations) having a comparatively small M-rank, including the A-rank of graphs in Subsection 1.1, the $\mathcal{H}$-rank and $\mathcal{S}$-rank of mixed graphs in Subsection 1.2 and the $\mathcal{A}$-rank of complex unit gain graphs in Subsection 1.3. Our main contribution is reported in Section 2 where we completely determine all graphs with $\mathrm{rank}_{A_\alpha}(G) = 2$. As a corollary, the graphs with $\mathrm{rank}_Q(G) = 2$ are also obtained.

1 A Brief Survey of Known Results

In this section, we review the results on the rank of matrices associated with ordinary graphs, oriented graphs, mixed graphs and complex unit gain graphs. Each of these matrices has rank 0 if and only if the corresponding graph is empty (i.e., has no edges).

1.1 *A*-rank of Graphs

Obviously, the A-rank of every non-empty graph is at least 2. In 1985, Constantine[5] determined the graphs with A-rank 2. Further results consider graphs whose A-rank is larger but close to 2.

Theorem 1.1[5] We have $\mathrm{rank}_A(G) = 2$ if and only if G is a complete bipartite graph with possible isolated vertices.

In 1999, Sciriha[34] determined the graphs with A-rank 3.

Theorem 1.2[34] We have $\mathrm{rank}_A(G) = 3$ if and only if G is a complete tripartite graph with possible isolated vertices.

To arrive at graphs with A-rank 4, we need some definitions. Let G be graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and let $\boldsymbol{m} = (m_1, m_2, \cdots, m_n)$ be a vector of positive integers. Denote by $G \bullet \mathbf{m}$ the graph obtained from G by replacing each vertex v_i of G with an independent set of m_i vertices $v_i^1, v_i^2, \cdots, v_i^{m_i}$ and joining v_i^s with v_j^t if and only if v_i and v_j are adjacent in G. The resulting graph $G \bullet \mathbf{m}$ is said to be obtained from G and $\boldsymbol{m}$ by a *vertex-multiplication*. For graphs $G_1, G_2, \cdots, G_k$, we denote by $\mathcal{M}(G_1, G_2, \cdots, G_k)$ the class of all graphs that can be obtained from one of the graphs in $\{G_1, G_2, \cdots, G_k\}$ by a vertex-multiplication.

Graphs with A-rank 4, have been considered in [11, 13, 17, 23, 37], and fully characterized by Chang et al.[1] and independently by Cheng and Liu[3].

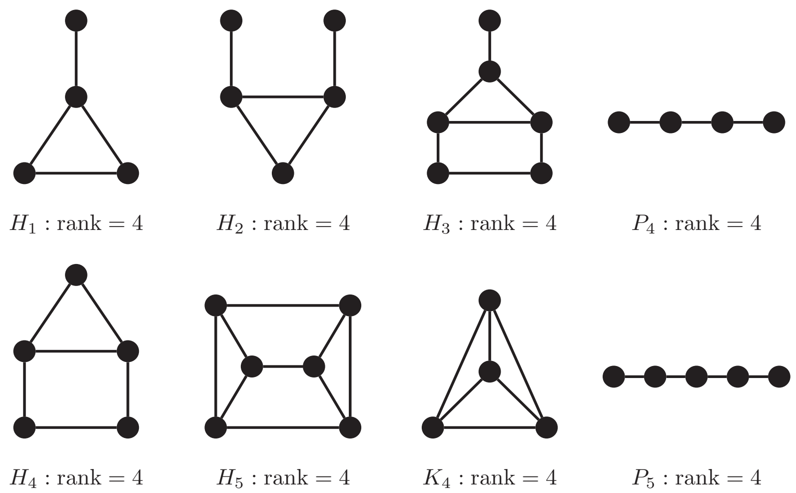

Figure 1 "Generators" of connected graphs with rank 4

Theorem 1.3[1, 3] For a connected graph G, $\mathrm{rank}_A(G) = 4$ holds if and only if $G \in \mathcal{M}(H_1, H_2, H_3, H_4, H_5, P_4, P_5, K_4)$, where the graphs H_1–H_5 are illustrated in Fig. 1.

Finally, graphs with A-rank 5 are considered in [13, 23, 25], and completely characterized by Chang et al.[2].

Theorem 1.4[2] For a connected graph G, $\mathrm{rank}_A(G) = 5$ holds if and only if $G \in \mathcal{M}(G_1, G_2, \cdots, G_{24})$, where the graphs G_1–G_{24} are illustrated in Fig. 2.

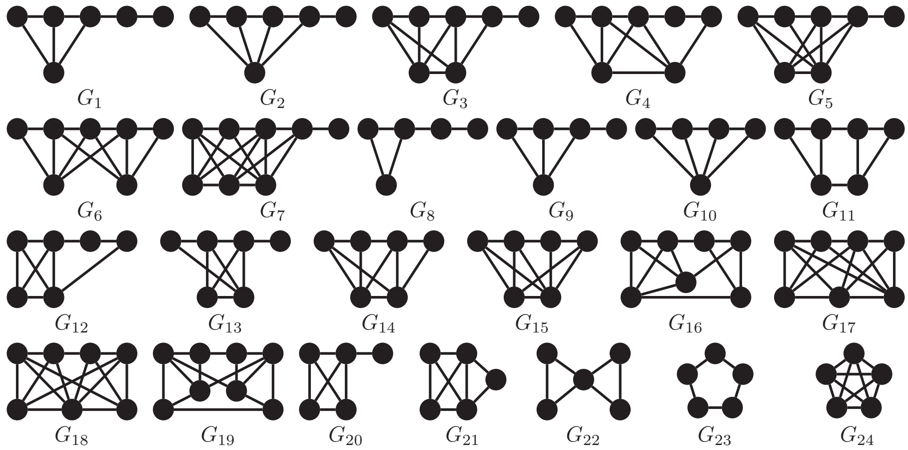

Figure 2 "Generators" of connected graphs with rank 5

1.2 $\mathcal{H}$-rank of Mixed Graphs

A *mixed graph* M is obtained from the underlying graph G by orienting a subset of its edges. The vertex-set $V(M)$ is the vertex-set of G, while the set of M is composed of the set $E_0(M)$ of undirected edges (or *digons*) and the set $E_1(M)$ of directed edges or *arcs*. The Hermitian *adjacency matrix* of a mixed graph M of order n is the $n \times n$ matrix $\mathcal{H}(M) = (h_{ij})$, where $h_{ij} = -h_{ji} = \mathrm{i}$ ($\mathrm{i} = \sqrt{-1}$) if there exists an arc orientated from i to j, $h_{ij} = h_{ji} = 1$ if there exists a digon (an edge between i and j), and $h_{kl} = 0$ otherwise.

The *converse* of a mixed graph M is the mixed graph $M^\intercal$ with the same vertex set, the same set of undirected edges and the arc-set $E_1(M^\intercal) = \{ij \mid ji \in E_1(M)\}$.

Suppose that the vertex-set of M is partitioned in four (possibly empty) sets, $V(M) = V_1 \cup V_{-1} \cup V_{\mathrm{i}} \cup V_{-\mathrm{i}}$. An edge $ij \in E(M)$ is said to be of type (a, b) for $a, b \in \{\pm 1, \pm \mathrm{i}\}$ if $i \in V_a$ and $j \in V_b$. A four-way switching with respect to the given partition is the operation that transforms M into the mixed graph M' in the following way:

(a) reversing the direction of all arcs of types $(1, -1)$, $(-1, 1)$, $(\mathrm{i}, -\mathrm{i})$, $(-\mathrm{i}, \mathrm{i})$;

(b) replacing each digon of type $(1, \mathrm{i})$ with a single arc directed from V_1 to V_{i} and replacing each digon of type $(-1, -\mathrm{i})$ with a single arc directed from V_{-1} to $V_{-\mathrm{i}}$;

(c) replacing each digon of type $(1, -\mathrm{i})$ with a single arc directed from $V_{-\mathrm{i}}$ to V_1;

(d) replacing each non-digon of type $(1, -\mathrm{i})$, $(-1, \mathrm{i})$, $(\mathrm{i}, 1)$ or $(-\mathrm{i}, -1)$ with a digon.

It is easy to see that the lower bound for the $\mathcal{H}$-rank of non-empty mixed graphs is 2. In 2016, the structure of a mixed graph M with $\mathcal{H}$-rank 2 was completely characterized by Mohar[30]. We say that mixed graphs M and M' are *switching equivalent* if one can be obtained from the other by a sequence of four-way switching and taking the converse. Accordingly, let $a \leq b \leq c$ be positive integers. Denote by $\vec{K}_3(a, b, c)$ the complete tripartite graph with parts A, B, C, where $|A| = a$, $|B| = b$ and $|C| = c$, and with all arcs from A to B, all arcs from B to C, and all arcs from C to A.

Theorem 1.5 [30, Theorem 5.8] Let M be a mixed graph of order n whose $\mathcal{H}$-rank is equal to 2 and let ρ be its positive eigenvalue. Then M is switching equivalent either to $K_{a,b} \cup tK_1$ or to $\vec{K}_3(a,b,c) \cup tK_1$, where $t \geq 0$. In the former case we have

$$n = a + b + t \quad \text{and} \quad \rho^2 = ab, \tag{1.1}$$

and in the latter case we have

$$n = a + b + c + t \quad \text{and} \quad \rho^2 = ab + bc + ac. \tag{1.2}$$

Conversely, for any $t \geq 0$ and $1 \leq a \leq b \leq c$, the mixed graphs $K_{a,b} \cup tK_1$ and $\vec{K}_3(a,b,c) \cup tK_1$ have $\mathcal{H}$-rank 2 and they satisfy (1.1) and (1.2), respectively.

Two vertices u, v in M are *twins* if M is switching equivalent to a mixed graph M' in which u and v have the same neighbourhood. Precisely, for each $w \in V(M')$, we have $vw \in E(M')$ if and only if $uw \in E(M')$, and $wv \in E(M')$ if and only if $wu \in E(M')$. By removing or adding twins the rank of the adjacency matrix remains the same (but the spectrum changes). "Being a twin" is an equivalence relation on $V(M)$. Let $[x]$ denote the equivalence class containing the vertex x. Mohar[30] defines the *twin reduction graph* of M, denoted by T_M, to be a graph whose vertices are the equivalence classes and $[x][y] \in E(T_M)$ if $xy \in E(M')$.

In 2017, Wang et al.[39] determined the mixed graphs M with $\mathcal{H}$-rank 3. A mixed triangle C is called *even triangle* if $\text{rank}_{\mathcal{H}}(C) = 3$. Clearly, T_M is an even triangle iff M is a complete tripartite graph.

Theorem 1.6 [39, Theorem 4.6] Let M be a connected mixed graph. Then $\text{rank}_{\mathcal{H}}(M) = 3$ if and only if its twin reduction T_M is an even triangle.

In 2021, Yang et al.[41] characterized all twin reduction graphs of connected bipartite mixed graphs with $\mathcal{H}$-rank 4, 6 or 8.

Theorem 1.7 [41, Theorem 3.3] Let $M[X,Y]$ be a connected bipartite mixed graph with partition (X,Y). Then $\text{rank}_{\mathcal{H}}(M[X,Y]) = 4$ if and only if the twin reduction graph is a connected subgraph induced by any k $(2 \leq k \leq 6)$ vertices of Y_2 and all two vertices of X_2 of $M_2[X_2, Y_2]$ shown in Fig. 3.

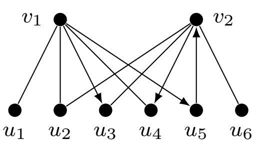

Figure 3 $M_2[X_2, Y_2]$

Theorem 1.8 [41, Theorem 3.4] Let $M[X,Y]$ be a connected bipartite mixed graph with partition (X,Y). Then $\text{rank}_{\mathcal{H}}(M[X,Y]) = 6$ if and only if the twin reduction graph is a connected subgraph induced by any k $(3 \leq k \leq 31)$ vertices of Y_4 and all three vertices of X_4 of $M[X_4, Y_4]$ shown in Fig. 4.

For two mixed bipartite graphs $M[X', Y'] = (V', E')$ and $M[X'', Y''] = (V'', E'')$, where $V' = X' \cup Y'$, $V'' = X'' \cup Y''$, and $X' = X''$, $Y' \cap Y'' = \emptyset$, the bipartite graph $M[X''', Y'''] = (V''', E''')$ may be regarded as a *vertex union graph* of the two bipartite graphs of $M[X', Y']$ and $M[X'', Y'']$, where $V''' = X''' \cup Y'''$ and $X''' = X' = X''$, $Y''' = Y' \cup Y''$, $E''' = E' \cup E''$.

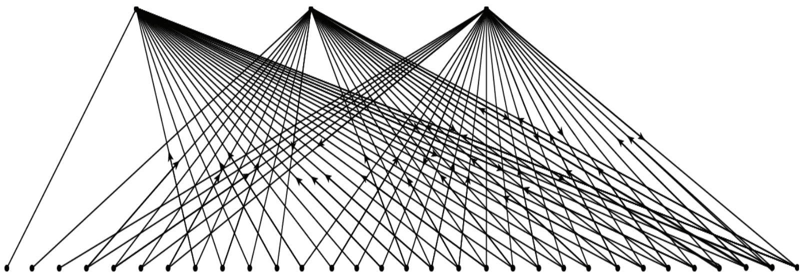

Figure 4 $M_4[X_4, Y_4]$

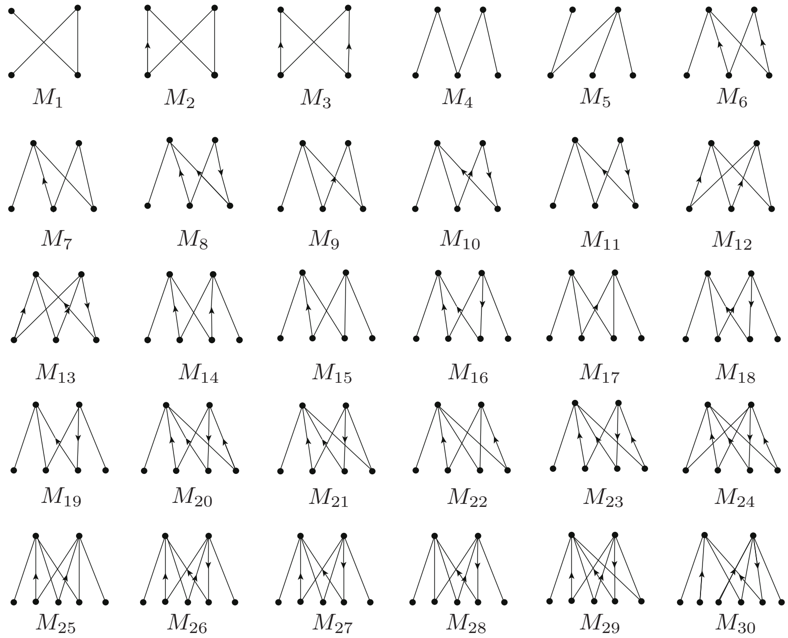

Figure 5 Twin reduction graphs $T_{M[X,Y]} = M_i$ with $\mathrm{rank}_{\mathcal{H}}(M_i) = 4$ for $1 \leq i \leq 30$

Theorem 1.9 [41, Theorem 3.5] Let $M[X, Y]$ be a connected bipartite mixed graph with partition (X, Y). Then $\mathrm{rank}_{\mathcal{H}}(M[X, Y]) = 8$ if and only if the twin reduction graph is the vertex union graph of any k $(4 \leq k \leq 150)$ mixed graphs of Fig. 5 such that the vertex union graph is connected.

The following problem arises naturally.

Problem 1.1 Identify all connected mixed graphs with $\mathcal{H}$-rank 4.

A particular class of mixed graphs are *oriented graphs* where we have $E_0(M) = \emptyset$. In other words, an oriented graph is a mixed graph without digons. An alternative definition says that an oriented graph $G^{\intercal}$ is a pair (G, σ) where G is the underlying graph and σ is the *orientation* that orients every edge from one vertex to the other. In this context, we usually deal with two matrices. The first one is the asymmetric *adjacency matrix* $S(G^{\sigma}) = (s_{ij})$ of G^{σ} defined by $s_{ij} = 1$ if $i \to j$ and $s_{ij} = 0$, otherwise[35]. Then the *S-rank* of G^{σ} is the rank of $SS^{\intercal}$. The vertex multiplication defined in the previous section is extended in a natural way to the framework of oriented graphs. Zhang et al.[46] have established a characterization of oriented graphs of S-rank at most 2.

Theorem 1.10[46, Lemma 2.14] The S-rank of a connected orientated graph G^σ is 1 if and only if G^σ is obtained from K_2^σ by a vertex-multiplication.

Theorem 1.11[46, Theorem 1.1] The S-rank of a connected oriented graph G^σ is 2 if and only if G^σ or $(G^\sigma)^\intercal$ is obtained from some of the oriented graphs of Fig. 6 by a vertex-multiplication.

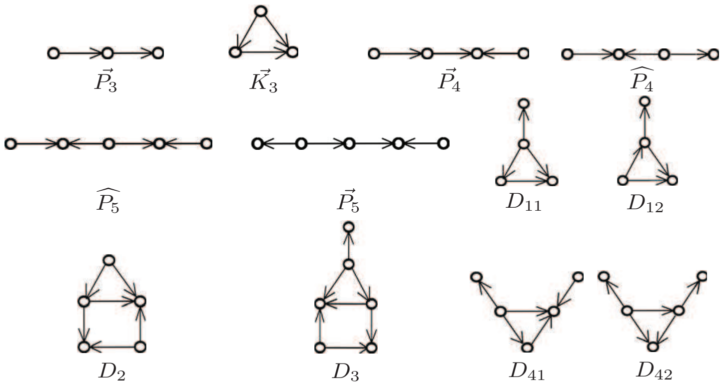

Figure 6 Oriented graphs with S-rank 2

The *skew adjacency matrix* $\mathcal{S}(G^\sigma) = (s_{ij})$ is a real skew symmetric matrix such that $s_{ij} = 1$ and $s_{ji} = -1$ if there is an arc from i to j and $s_{ij} = s_{ji} = 0$, otherwise. The $\mathcal{S}$-rank or the *skew-rank* is the rank of $\mathcal{S}(G^\sigma)$. Since the non-zero eigenvalues of $\mathcal{S}(G^\sigma)$ come as complex conjugates, we have that the skew-rank of every oriented graph is even. One may observe that if G^σ is interpreted as a mixed graph with adjacency matrix $\mathcal{H}(G^\sigma)$, then $\mathcal{H}(G^\sigma) = \mathrm{i}\mathcal{S}(G^\sigma)$, and therefore many results concerning the rank of the corresponding matrix can be transferred from the domain of mixed graphs. We have the following one in particular.

Theorem 1.12[19] Let G be an oriented connected graph. Then $\mathrm{rank}_{\mathcal{S}}(G^\sigma) = 2$ if and only if G is a complete multipartite graph.

The remaining results concerning the rank of the skew symmetric matrix are mostly focused on oriented graphs with a restricted number of cycles. For the trees, Li and Yu[24] obtained the following characterization.

Theorem 1.13[24] For an oriented tree T^σ with a matching number $\beta(T)$, we have

$$\mathrm{rank}_{\mathcal{S}}(T^\sigma) = \mathrm{rank}_A(T) = 2\beta(T).$$

For unicyclic and bicyclic oriented graphs with a comparatively small skew rank we refer the reader to [32, 33] and [27], respectively.

We conclude this section with another open problem, in fact a particular case of Problem 1.1.

Problem 1.2 Identify all oriented graphs with $\mathcal{S}$-rank 4.

1.3 $\mathcal{A}$-rank of Complex Unit Gain Graphs

Let $\mathbb{T}$ be the group of all complex numbers z with $|z| = 1$. A *complex unit gain graph*, or simply a $\mathbb{T}$-*gain graph*, is a triple $\Phi = (\Gamma, \mathbb{T}, \varphi)$ consisting of a graph $\Gamma = (V_\Gamma, E_\Gamma)$, the circle group $\mathbb{T}$ and a gain function $\varphi : \vec{E}_\Gamma \to \mathbb{T}$ such that $\varphi(v_iv_j) = \varphi(v_jv_i)^{-1}$ for any edge $v_iv_j \in E_\Gamma$, where $\vec{E} = \{uv, vu \mid uv \in E_\Gamma\}$ is the set of oriented edges. It is well known that the concept of $\mathbb{T}$-gain graphs is an extension of simple graphs, signed graphs and mixed graphs.

The *adjacency matrix* $\mathcal{A}(\Phi) = (a_{ij}) \in \mathbb{C}^{n\times n}$ of a $\mathbb{T}$-gain graph $\Phi = (\Gamma, \mathbb{T}, \varphi)$ with n vertices is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}), & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

If v_i is adjacent to v_j, then $a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})} = \overline{a_{ji}}$.

For the $\mathbb{T}$-gain graphs with at least one edge, it is easy to see that the $\mathcal{A}$-rank is at least 2. Yu et al.[45] studied the unicyclic complex unit gain graphs with rank 2. Lu et al.[26] investigated the bicyclic complex unit gain graphs with rank 2. Xu et al.[40] entirely determined the classification of $\mathbb{T}$-gain graphs with rank 2.

For a complete bipartite graph $K_{m,l}$ with vertex set partitioned as $X \cup Y$, let φ be the gain function such that $\varphi(xy) = 1$ for all $x \in X$, $y \in Y$ and we call $\Psi = (K_{m,l}, \mathbb{T}, \varphi)$ endowed with *balanced gain*. For a complete tripartite graph $K_{r,s,t}$ with vertex set partitioned as $X \cup Y \cup Z$. Let τ be the gain function such that $\tau(xy) = \tau(xz) = 1$ and $\tau(yz) = \mathrm{i}$ for all $x \in X, y \in Y, z \in Z$. If the gain function on $K_{r,s,t}$ is taken in such a way we call $\Gamma = (K_{r,s,t}, \mathbb{T}, \tau)$ endowed with *quasi-balanced gain*.

Theorem 1.14[40, Theorem 4.6] Let $\Phi = (G, \mathbb{T}, \varphi)$ be a connected $\mathbb{T}$-gain graph. Then $\operatorname{rank}_{\mathcal{A}}(\Phi) = 2$ if and only if Φ is either switching equivalent to $\Psi = (K_{m,l}, \mathbb{T}, \psi)$ or switching equivalent to $\Gamma = (K_{r,s,t}, \mathbb{T}, \tau)$, where $\Psi = (K_{m,l}, \mathbb{T}, \psi)$ is a $\mathbb{T}$-gain complete bipartite graph with balanced gain and $\Gamma = (K_{r,s,t}, \mathbb{T}, \tau)$ is a $\mathbb{T}$-gain complete tripartite graph with quasi-balanced gain.

Let C_n^φ be a weighted cycle with vertex set $\{v_1, v_2, \cdots, v_k\}$ such that $v_iv_{i+1} \in E(C_n^\varphi)$ $(1 \le i \le k-1)$, $v_1v_k \in E(C_n^\varphi)$. Let $w_i = \varphi(v_iv_{i+1})$ and $w_k = \varphi(v_kv_1)$. A $\mathbb{T}$-gain even cycle C_n^φ is said to be of *Type A* if $(-1)^{\frac{n}{2}} w_n = w_1w_2w_3\cdots w_{n-2}w_{n-1}$; A $\mathbb{T}$-gain odd cycle C_n^φ is said to be of *Type C* if $\mathrm{Re}((-1)^{\frac{n-1}{2}} w_1w_2w_3\cdots w_{n-2}w_{n-1}\bar{w}_n) > 0$; C_n^φ is said to be of *Type D* if $\mathrm{Re}((-1)^{\frac{n-1}{2}} w_1w_2w_3\cdots w_{n-2}w_{n-1}\bar{w}_n) < 0$; C_n^φ is said to be of *Type E* if $\mathrm{Re}((-1)^{\frac{n-1}{2}} w_1w_2w_3\cdots w_{n-2}w_{n-1}\bar{w}_n) = 0$, where $\mathrm{Re}(\,\cdot\,)$ is the real part of a complex number.

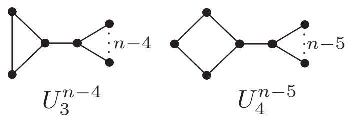

Figure 7 Two unicyclic graphs U_3^{n-4} and U_4^{n-5}

In 2015, Yu et al.[45] identified the $\mathbb{T}$-gain unicyclic graphs with rank 3.

Theorem 1.15[45, Theorem 14] Let G^φ be a $\mathbb{T}$-gain unicyclic graph of order n. Then $\operatorname{rank}_{\mathcal{A}}(G^\varphi) = 3$ if and only if G^φ is one of the following $\mathbb{T}$-gain cycles: C_3^φ of Type C or Type D.

By [45, Theorems 9–12], some unicyclic $\mathbb{T}$-gain graphs with rank 4 are characterized.

Theorem 1.16 Let G^φ be a $\mathbb{T}$-gain unicyclic graph of order n. If G^φ is C_5^φ of Type E, or C_6^φ of Type A, or $\mathbb{T}$-gain graphs with U_3^{n-4} (Fig. 7) as their underlying graphs in which the cycle C_3^{φ} is of Type E, or $\mathbb{T}$-gain graphs with U_4^{n-5} (Fig. 7) as their underlying graphs in which the cycle C_4^{φ} is of Type A, then $\mathrm{rank}_{\mathcal{A}}(G^{\varphi}) = 4$.

In 2017, Lu et al.[26] determined the bicyclic $\mathbb{T}$-gain graphs with $\mathcal{A}$-rank 3 or 4.

Theorem 1.17[26, Theorem 4.4] Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain connected bicyclic graph of order $n(\geq 4)$ with $\mathcal{A}$-rank 3 or 4. Then if Φ has no pendant vertices, then $\mathrm{rank}_{\mathcal{A}}(\Phi) = 3, 4$ with the graphs being given in Table 1; if Φ has pendant vertices, then $\mathrm{rank}_{\mathcal{A}}(\Phi) = 4$, where $n \geq 5$ and Φ has one of the graphs in Fig. 8 as an induced subgraph, which are given in Table 2.

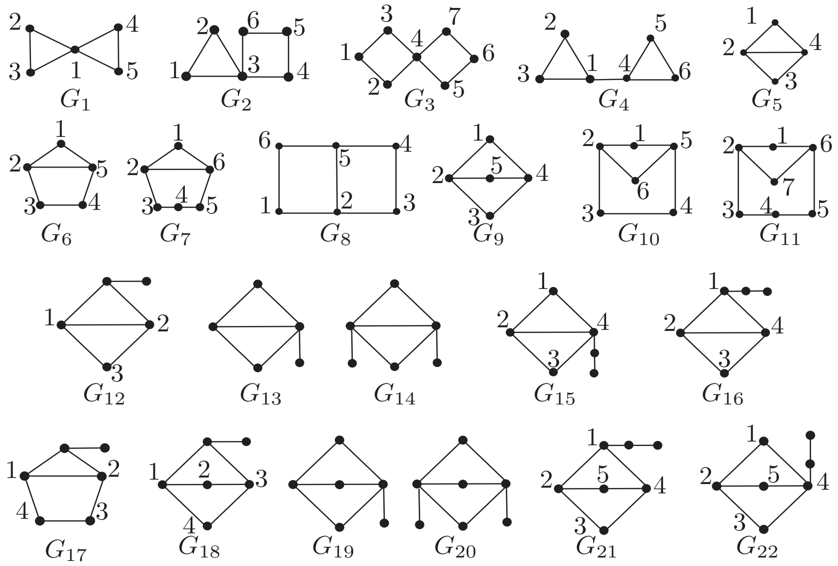

Figure 8 The graphs G_1–G_{22}

If the set of gains for a $\mathbb{T}$-gain graph is $\{\pm 1\}$, then Φ can be viewed as the well known signed graph. Lu et al.[28] studied how about the relations between the rank of an unbalanced signed graph and the rank of its underlying graph.

We propose the following problem to end this section.

Problem 1.3 For the $\mathbb{T}$-gain graphs with $\mathcal{A}$-rank 3 or 4,

(i) determine all the unicyclic $\mathbb{T}$-gain connected graphs with $\mathcal{A}$-rank 4;

(ii) give a complete characterization of $\mathbb{T}$-gain connected graphs with $\mathcal{A}$-rank 3 or 4.

2 A_α-rank of Graphs

In this section we will fully determine the graphs with A_α-rank 2. In Subsection 2.1 we investigate the properties of A_α with rank 2; and in Subsection 2.2 we give a complete characterization of graphs with $\mathrm{rank}_{A_\alpha}(G) = 2$.

2.1 Properties of A_α with Rank 2

We use n to denote the order (i.e., the number of vertices) of a graph. Two rows $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ of a matrix are equal if they are equal considered as real vectors, i.e., if $x_i = y_i$ for $1 \leq i \leq n$. We start with the following lemma.

Table 1 The gain conditions for each gain graph in Theorem 1.17 with rank 3 or 4

Rank	Gain graph G^φ	Gain conditions of G^φ
3	G_5^φ	The subgraph induced on vertices $1,2,4$ is of Type C or D and the subgraph induced on vertices $1,2,3,4$ is of Type A.
4	G_1^φ	$\mathrm{Re}(\varphi(v_1v_3v_2v_1)) + \mathrm{Re}(\varphi(v_1v_5v_4v_1)) = 0$.
	G_2^φ	The subgraph induced on vertices $3,4,5,6$ is of Type A and the subgraph induced on vertices $1,2,3$ is of Type E.
	G_3^φ	The subgraphs induced on vertices $1,2,4,3$ and vertices $4,5,6,7$ are of Type A.
	G_5^φ	The subgraph induced on vertices $1,2,4$ is of Type C, or D, or E and the subgraph induced on vertices $1,2,3,4$ is of Type B.
	G_6^φ	$\mathrm{Re}(\varphi(v_1v_2v_5v_1)) = \mathrm{Re}(\varphi(v_1v_2v_3v_4v_5v_1))$.
	G_7^φ	The subgraph induced on vertices $1,2,6$ is of Type E and the subgraph induced on vertices $1,2,3,4,5,6$ is of Type A.
	G_8^φ	$\varphi(v_1v_2v_3v_4v_5v_6v_1) - \varphi(v_1v_2v_5v_6v_1) + 1 = 0$.
	G_9^φ	The subgraph induced on vertices $1,2,3,4$ is of Type A and the subgraph induced on vertices $1,2,5,4$ is of Type B or the subgraph induced on vertices $1,2,3,4$ is of Type B and the subgraph induced on vertices $1,2,5,4$ is of Type A or B.
	G_{10}^φ	The subgraph induced on vertices $1,2,6,5$ is of Type A and the subgraph induced on vertices $1,2,3,4,5$ is of Type E.
	G_{11}^φ	The subgraphs induced on vertices $1,2,7,6$ and vertices $1,2,3,4,5,6$ are of Type A.

Table 2 The gain conditions for each gain graph in Theorem 1.17 with $\mathbf{rank}_{\boldsymbol{A}}(\boldsymbol{\Phi}) = 4$

Gain graphs G^φ	Gain conditions of G^φ
G_{12}^φ	The subgraph induced on vertices $1,2,3$ is of Type E.
$G_{13}^\varphi, G_{14}^\varphi, G_{19}^\varphi, G_{20}^\varphi$	Any gain.
$G_{15}^\varphi, G_{16}^\varphi$	The subgraph induced on vertices $1,2,4$ is of Type E and the subgraph induced on vertices $1,2,3,4$ is of Type A.
$G_{17}^\varphi, G_{18}^\varphi$	The subgraph induced on vertices $1,2,3,4$ is of Type A.
$G_{21}^\varphi, G_{22}^\varphi$	The subgraphs induced on vertices $1,2,3,4$ and vertices $1,2,5,4$ are of Type A.

Lemma 2.1 For every graph G, any two non-equal non-zero rows in $A_\alpha(G)$ are linearly independent.

Proof The statement is obvious for $\alpha = 1$, since A_1 is a diagonal matrix, and for $\alpha = 0$, since in the adjacency matrix every non-zero element is 1.

Let further $\alpha \in (0,1)$. The ith diagonal entry of A_α has the form $a_{ii} = \alpha d_i$ for $1 \leq i \leq n$. Assume by way of contradiction that $\boldsymbol{x} = (a_{i1}, a_{i2}, \cdots, a_{in})$ and $\boldsymbol{y} = (a_{j1}, a_{j2}, \cdots, a_{jn})$ are non-equal non-zero rows in A_α that are linearly dependent. We may suppose that $d_i \geq d_j \geq 1$. There exists $h > 0$ such that $\boldsymbol{y} = h\boldsymbol{x}$.

If $d_i \geq 2$, then at least two entries in $\boldsymbol{x}$ are $1 - \alpha$, and then at least two entries in $\boldsymbol{y}$ are $h(1-\alpha) \neq 0$. Since $a_{j_k} \in \{0, 1-\alpha\}$ for every k distinct from j, we find that $h(1-\alpha) = \alpha$.

Hence, $h = 1$, which means that $\boldsymbol{x} = \boldsymbol{y}$, a contradiction.

If $d_i = 1$, then we also have $d_j = 1$, and then $a_{ii} = \alpha d_i = \alpha$ and $a_{jj} = \alpha d_j = \alpha$. From $\boldsymbol{y} = h\boldsymbol{x}$, we get $a_{ji} = ha_{ii} = h\alpha \neq 0$. Since $a_{ji} \in \{0, 1-\alpha\}$, we have $a_{ji} = 1-\alpha$, which together with the previous equality leads to $h\alpha = 1-\alpha$. We also have $\alpha = a_{jj} = ha_{ij} = ha_{ji} = 1-\alpha$, which together with the previous equality gives $h = 1$, i.e., $\boldsymbol{x} = \boldsymbol{y}$, a contradiction. $\square$

In what follows we give a closer characterization of A_α having rank 2. It is convenient to label the vertices of G in such a way that the rows of A_α are indexed in non-increasing order by their sums. This, in particular, means that we have $d_1 \geq d_2 \geq \cdots \geq d_n$.

Evidently, if $\mathrm{rank}(A_\alpha) = 2$, then A_α has at least two distinct non-zero rows. Without loss of generality, let $\boldsymbol{x} = (a_{1,1}, a_{1,2}, \cdots, a_{1,n})$ be the first row of A_α, and $\boldsymbol{y} = (a_{r_1,1}, a_{r_1,2}, \cdots, a_{r_1,n})$ with $r_1 \geq 2$, be the first row which is non-equal to $\boldsymbol{x}$. By Lemma 2.1, $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent. Let $\boldsymbol{z} = (a_{r_2,1}, a_{r_2,2}, \cdots, a_{r_2,n})$ be the first row such that $\boldsymbol{z} \notin \{\boldsymbol{x}, \boldsymbol{y}\}$. Accordingly, A_α has the following form:

$$A_\alpha = \left[\begin{array}{cccc}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{r_1-1,1} & a_{r_1-1,2} & \cdots & a_{r_1-1,n} \\
a_{r_1,1} & a_{r_1,2} & \cdots & a_{r_1,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{r_2-1,1} & a_{r_2-1,2} & \cdots & a_{r_2-1,n} \\
a_{r_2,1} & a_{r_2,2} & \cdots & a_{r_2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{r_3-1,1} & a_{r_3-1,2} & \cdots & a_{r_3-1,n} \\
\hline
\multicolumn{4}{c}{*}
\end{array}\right] = \left[\begin{array}{c}
x \\ \vdots \\ x \\ y \\ \vdots \\ y \\ z \\ \vdots \\ z \\ \hline *
\end{array}\right], \tag{2.1}$$

where the first $(r_1 - 1)$ rows coincide with $\boldsymbol{x}$, the rows from r_1th to $(r_2 - 1)$th are $\boldsymbol{y}$ and the rows from r_2th to $(r_3 - 1)$th are $\boldsymbol{z}$ (with $r_3 - 1 \leq n$). Let F_{11}, F_{22}, F_{33} be

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,r_1-1} \\ \vdots & \ddots & \vdots \\ a_{r_1,1} & \cdots & a_{r_1-1,r_1-1} \end{bmatrix}, \quad \begin{bmatrix} a_{r_1,r_1} & \cdots & a_{r_1,r_2-1} \\ \vdots & \ddots & \vdots \\ a_{r_2-1,r_1} & \cdots & a_{r_2-1,r_2-1} \end{bmatrix}, \quad \begin{bmatrix} a_{r_2,r_2} & \cdots & a_{r_2,r_3-1} \\ \vdots & \ddots & \vdots \\ a_{r_3-1,r_2} & \cdots & a_{r_3-1,r_3-1} \end{bmatrix},$$

respectively, denoting the diagonal blocks of A_α. Then A_α can be rewritten as

$$\left[\begin{array}{c}
x \\ \vdots \\ x \\ \hline y \\ \vdots \\ y \\ \hline z \\ \vdots \\ z \\ \hline *
\end{array}\right] = \left[\begin{array}{ccc|ccc|ccc|c}
a_{1,1} & \cdots & a_{1,r_1-1} & a_{1,r_1} & \cdots & a_{1,r_2-1} & a_{1,r_2} & \cdots & a_{1,r_3-1} & \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
a_{r_1-1,1} & \cdots & a_{r_1-1,r_1-1} & a_{r_1-1,r_1} & \cdots & a_{r_1-1,r_2-1} & a_{r_1-1,r_2} & \cdots & a_{r_1-1,r_3-1} & \\
\cline{1-9}
a_{r_1,1} & \cdots & a_{r_1,r_1-1} & a_{r_1,r_1} & \cdots & a_{r_1,r_2-1} & a_{r_1,r_2} & \cdots & a_{r_1,r_3-1} & \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & * \\
a_{r_2-1,1} & \cdots & a_{r_2-1,r_1-1} & a_{r_2-1,r_1} & \cdots & a_{r_2-1,r_2-1} & a_{r_2-1,r_2} & \cdots & a_{r_2-1,r_3-1} & \\
\cline{1-9}
a_{r_2,1} & \cdots & a_{r_2,r_1-1} & a_{r_2,r_1} & \cdots & a_{r_2,r_2-1} & a_{r_2,r_2} & \cdots & a_{r_2,r_3-1} & \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
a_{r_3-1,1} & \cdots & a_{r_3-1,r_1-1} & a_{r_3-1,r_1} & \cdots & a_{r_3-1,r_2-1} & a_{r_3-1,r_2} & \cdots & a_{r_3-1,r_3-1} & \\
\hline
\multicolumn{9}{c|}{*} & *
\end{array}\right]$$

$$= \left[\begin{array}{ccc|c} F_{11} & F_{12} & F_{13} & \\ F_{21} & F_{22} & F_{23} & B \\ F_{31} & F_{32} & F_{33} & \\ \hline & B^{\mathsf{T}} & & D \end{array}\right]. \tag{2.2}$$

In what follows we compute the blocks $F_{i,j}$ for $1 \leq i, j \leq 3$.

If $r_1 - 1 = 1$, then $F_{11} = a_{1,1}$. For otherwise, we have $a_{1,1} = a_{2,1} = \cdots = a_{r_1-1,1}$. Since $a_{2,1} = a_{3,1} = \cdots = a_{r_1-1,1} \in \{0, 1-\alpha\}$ and $a_{1,1} \neq 0$, we find that $a_{1,1} = a_{2,1} = \cdots = a_{r_1-1,1} = 1-\alpha$. By the symmetry of A_α, we also have $a_{1,2} = a_{1,3} = \cdots = a_{1,r_1-1} = 1-\alpha$. Since the first $r_1 - 1$ rows coincide with $\boldsymbol{x}$, we conclude that, for $r_1 - 1 \geq 2$,

$$F_{11} = (1-\alpha)J_{(r_1-1)\times(r_1-1)}. \tag{2.3}$$

In a very similar way, we get

$$F_{22} = (1-\alpha)J_{(r_2-1)\times(r_2-1)} \tag{2.4}$$

for $r_2 - 1 \geq 2$.

Consider now F_{12} and F_{21}. If $a_{r_1,1} = a_{1,r_1} = 1-\alpha$, then $a_{r_1,1} = a_{r_1+1,1} = \cdots = a_{r_2-1,1} = 1-\alpha$ by (2.1). Thus $a_{1,r_1} = a_{1,r_1+1} = \cdots = a_{1,r_2-1} = 1-\alpha$ by the symmetry of A_α. Using again the assumption that the first $r_1 - 1$ rows coincide with $\boldsymbol{x}$, we obtain $a_{i,r_1} = a_{i,r_1+1} = \cdots = a_{i,r_2-1} = 1-\alpha$, where $1 \leq i \leq r_1 - 1$, i.e., $F_{12} = (1-\alpha)J_{(r_1-1)\times(r_2-r_1)}$. If $a_{r_1,1} = a_{1,r_1} = 0$, in the same way we find $F_{12} = O_{(r_1-1)\times(r_2-r_1)}$. Therefore,

$$F_{12} = a_{1,r_1}J_{(r_1-1)\times(r_2-r_1)}. \tag{2.5}$$

By the symmetry, we get $F_{21} = a_{r_1,1}J_{(r_2-r_1)\times(r_1-1)}$.

If $\boldsymbol{z} \neq 0$, then $F_{33} \neq O$ by the identity (2.2). Following the computation for F_{11}, we arrive at

$$F_{33} = (1-\alpha)J_{(r_3-1)\times(r_3-1)} \tag{2.6}$$

for $r_3 - 1 \geq 2$. As before, we obtain

$$F_{13} = a_{1,r_2}J, \quad F_{23} = a_{r_1,r_2}J, \quad F_{31} = a_{r_2,1}J, \quad F_{32} = a_{r_2,r_1}J. \tag{2.7}$$

Now, we are ready to prove the following propositions. In the proofs we continue to use the previous notation.

Proposition 2.1 Suppose that $\operatorname{rank}(A_\alpha) = 2$. If $a_{1,1}, a_{r_1,r_1}, a_{r_2,r_2}$ are distinct from 0, then at most one of a_{1,r_1}, a_{1,r_2}, a_{r_1,r_2} is 0.

Proof Since $a_{r_2,r_2} = \alpha d_{r_2} \neq 0$, we have $\boldsymbol{z} = (a_{r_2,1}, a_{r_2,2}, \cdots, a_{r_2,n}) \neq \mathbf{0}$. Since $\operatorname{rank}(A_\alpha) = 2$, we have that each of the rows $\boldsymbol{x}$, $\boldsymbol{y}$ or $\boldsymbol{z}$ is a linear combination of the remaining two. Together with (2.2) and the assumption of this proposition, this leads to the conclusion that none of $(a_{r_1,1}, a_{r_2,1})$, (a_{1,r_1}, a_{r_2,r_1}) and (a_{1,r_2}, a_{r_1,r_2}) is $(0,0)$. Since A_α is symmetric, we also have $a_{r_1,1} = a_{1,r_1}$, $a_{r_2,1} = a_{1,r_2}$ and $a_{r_1,r_2} = a_{r_2,r_1}$, which leads to the conclusion that at most one of a_{1,r_1}, a_{1,r_2}, a_{r_1,r_2} is 0. □

Proposition 2.2 If $\operatorname{rank}(A_\alpha) = 2$, then A_α has at most three distinct non-zero rows.

Proof With the previous notation, assume that there exists a row

$$\boldsymbol{t} = (a_{r_3,1}, a_{r_3,2}, \cdots, a_{r_3,n}) = p_1\boldsymbol{x} + q_1\boldsymbol{y} = p_2\boldsymbol{x} + q_2\boldsymbol{z} = p_3\boldsymbol{y} + q_3\boldsymbol{z}, \tag{2.8}$$

where $\boldsymbol{t} \notin \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{0}\}$. Thus, for a_{r_3,r_3} we have

$$a_{r_3,r_3} = p_1a_{1,r_3} + q_1a_{r_1,r_3} = p_2a_{1,r_3} + q_2a_{r_2,r_3} = p_3a_{r_1,r_3} + q_3a_{r_2,r_3}. \tag{2.9}$$

From $\boldsymbol{t} \neq \boldsymbol{0}$, we get $a_{r_3,r_3} = \alpha d_{r_3} \neq 0$. From $a_{r_3,1} = a_{1,r_3}$, $a_{r_1,r_3} = a_{r_3,r_1}$ and $a_{r_2,r_3} = a_{r_3,r_2}$, we get that at most one of $a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}$ is 0, by (2.9); for otherwise, we would have the impossible scenario $a_{r_3,r_3} = 0$.

Now, we distinguish two cases depending on $a_{1,r_1}, a_{1,r_2}, a_{r_1,r_2}$.

Case 1: $a_{1,r_1} = a_{1,r_2} = a_{r_1,r_2} = 1 - \alpha$.

Suppose first that exactly one of $a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}$ is 0, say $(a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}) = (1-\alpha, 1-\alpha, 0)$. From

$$1 - \alpha = a_{r_3,1} = p_2a_{1,1} + q_2a_{r_2,1} = p_2a_{1,1} + q_2(1-\alpha)$$

and

$$1 - \alpha = a_{r_3,r_1} = p_2a_{1,r_1} + q_2a_{r_2,r_1} = p_2(1-\alpha) + q_2(1-\alpha)$$

we obtain $a_{1,1} = 1 - \alpha$. On the other hand,

$$0 = a_{r_3,r_2} = p_1a_{1,r_2} + q_1a_{r_1,r_2} = p_1(1-\alpha) + q_1(1-\alpha)$$

and

$$1 - \alpha = a_{r_3,1} = p_1a_{1,1} + q_1a_{r_1,1} = p_1(1-\alpha) + q_1(1-\alpha)$$

lead to the contradiction $0 = 1 - \alpha$. In the same way we eliminate the two more possibilities: $(a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}) \in \{(0, 1-\alpha, 1-\alpha), (1-\alpha, 0, 1-\alpha)\}$.

Suppose now that $(a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}) = (1-\alpha, 1-\alpha, 1-\alpha)$. From

$$1 - \alpha = a_{r_3,1} = p_2a_{1,1} + q_2a_{r_2,1} = p_2a_{1,1} + q_2(1-\alpha)$$

and

$$1 - \alpha = a_{r_3,r_1} = p_2a_{1,r_1} + q_2a_{r_2,r_1} = p_2(1-\alpha) + q_2(1-\alpha)$$

we obtain $\alpha d_1 = a_{1,1} = 1 - \alpha$. Analogously, $\alpha d_{r_1} = 1 - \alpha$ and $\alpha d_{r_2} = 1 - \alpha$. Thus, $d_1 = d_{r_1} = d_{r_2} = \frac{1}{\alpha} - 1$. Inserting this into the identity (2.2) and taking into account (2.3)–(2.8), we obtain

$$A_\alpha = \begin{bmatrix} (1-\alpha)J_{\frac{1}{\alpha}\times\frac{1}{\alpha}} & O \\ O & O \end{bmatrix},$$

but this yields $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z} = \boldsymbol{t}$, a contradiction.

Case 2: Exactly one of $a_{1,r_1}, a_{1,r_2}, a_{r_1,r_2}$ is 0.

Without loss of generality, we can take that $a_{1,r_1} = a_{1,r_2} = 1-\alpha$ and $a_{r_1,r_2} = 0$. We already know that at most one of $a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}$ is 0. In what follows we consider the possibilities that arise from this.

Suppose first that $(a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}) = (1-\alpha, 1-\alpha, 0)$. Here we have

$$0 = a_{r_3,r_2} = p_1 a_{1,r_2} + q_1 a_{r_1,r_2} = p_1(1-\alpha) + q_1 0.$$

Since $1-\alpha \neq 0$, there must be $p_1 = 0$, and then we have $\boldsymbol{t} = q_1\boldsymbol{y}$ (by (2.9)). Since $\boldsymbol{t} \neq \boldsymbol{0}$, we have $q_1 \neq 0$, which implies $\boldsymbol{t} = \boldsymbol{y}$ as, by Lemma 2.1, linearly dependent non-zero rows must be equal. The latter contradicts the initial assumption that $\boldsymbol{t} \neq \boldsymbol{y}$.

Suppose now that $(a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}) = (1-\alpha, 0, 1-\alpha)$. As before, we get

$$0 = a_{r_3,r_1} = p_2 a_{1,r_1} + q_2 a_{r_2,r_1} = p_2(1-\alpha) + q_2 0.$$

Since $1-\alpha \neq 0$, there must be $p_2 = 0$, which gives $\boldsymbol{t} = q_2\boldsymbol{z}$. The inequality $\boldsymbol{t} \neq \boldsymbol{0}$ yields $q_2 \neq 0$, which implies $\boldsymbol{t} = \boldsymbol{z}$, which is a contradiction as before.

Finally, we suppose that $(a_{r_3,1}, a_{r_3,r_1}, a_{r_3,r_2}) = (a_{r_3,1}, 1-\alpha, 1-\alpha)$, where $a_{r_3,1} \in \{1-\alpha, 0\}$ and compute

$$1-\alpha = a_{r_3,r_2} = p_1 a_{1,r_2} + q_1 a_{r_1,r_2} = p_1(1-\alpha)$$

and

$$1-\alpha = a_{r_3,r_1} = p_1 a_{1,r_1} + q_1 a_{r_1,r_1} = p_1(1-\alpha) + q_1 a_{r_1,r_1}.$$

It follows that $q_1 a_{r_1,r_1} = 0$. Since $a_{r_1,r_1} = \alpha d_{r_1} \neq 0$, we get $q_1 = 0$, which gives $\boldsymbol{t} = p_1\boldsymbol{x}$. From $\boldsymbol{t} \neq \boldsymbol{0}$, we get $p_1 \neq 0$, which implies $\boldsymbol{t} = \boldsymbol{x}$, a contradiction. □

Proposition 2.3 For a graph G, if $\operatorname{rank}(A(G)) \neq 2$ and $\operatorname{rank}(A_\alpha(G)) = 2$, then $2 \leq \frac{1}{\alpha} \in \{d_1+1, d_{r_1}+1, d_{r_2}+1\}$, with above notation for vertex degrees.

Proof We continue to use the notation introduced at the beginning of this section and distinguish the following two cases depending on the occurrences of $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ in A_α.

Case 1: Each of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ occurs at most one time in A_α. (We will see that this case is rejected by the assumptions of the statement.)

First, the case in which $\boldsymbol{x}$ occurs one time and the remaining rows are $\boldsymbol{0}$ is impossible since $\boldsymbol{x} \neq \boldsymbol{0}$ and A_α is symmetric.

Next, if $\boldsymbol{x}$ occurs exactly one time, $\boldsymbol{y}$ occurs exactly one time and the remaining rows are $\boldsymbol{0}$, then $G \cong K_2 \cup (n-2)K_1$ by the identity (2.2). This yields $\operatorname{rank}(A(G)) = 2$, a contradiction.

Finally, let $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ occur one time each. From (2.1), we have

$$A_\alpha = \left[\begin{array}{ccc|c} a_{1,1} & a_{1,2} & a_{1,3} & \\ a_{2,1} & a_{2,2} & a_{2,3} & O \\ a_{3,1} & a_{3,2} & a_{3,3} & \\ \hline & O & & O \end{array}\right] = \left[\begin{array}{c} x \\ y \\ z \\ \hline 0 \end{array}\right].$$

We know from Proposition 2.2 that at most one of $a_{1,2}, a_{1,3}, a_{2,3}$ is 0. Suppose first that none of them is 0, i.e., $a_{1,2} = a_{1,3} = a_{2,3} = 1-\alpha$. Since $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly dependent, we have

$$\det \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \det \begin{bmatrix} 2\alpha & 1-\alpha & 1-\alpha \\ 1-\alpha & 2\alpha & 1-\alpha \\ 1-\alpha & 1-\alpha & 2\alpha \end{bmatrix} = (3\alpha-1)^2 = 0.$$

Hence, $\alpha = \frac{1}{3}$, but this leads to $\operatorname{rank}(A_\alpha) = 1$.

Suppose now that exactly one of the mentioned entries is 0. We are allowed to say that $a_{1,2} = a_{1,3} = 1-\alpha$ and $a_{2,3} = 0$. Thus $G \cong K_{1,2} \cup (n-3)K_1$ and

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & \\ 1 & 0 & 0 & O \\ 1 & 0 & 0 & \\ \hline & O & & O \end{array}\right],$$

but this leads to $\operatorname{rank}(A) = 2$, a contradiction.

Case 2: At least one of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ occurs at least two times in A_α.

If $\boldsymbol{x}$ occurs at least two times, then we have $a_{2,1} = a_{1,1} = \alpha d_1$. From $\alpha d_1 \neq 0$ and $a_{2,1} \in \{0, 1-\alpha\}$, we obtain $\alpha d_1 = 1-\alpha$, which gives $\frac{1}{\alpha} = d_1 + 1$. We have $2 \leq \frac{1}{\alpha}$ because $d_1 \geq 1$.

If $\boldsymbol{x}$ occurs one time and $\boldsymbol{y}$ occurs at least two times, then $a_{r_1,r_1} = a_{r_1+1,r_1} = \alpha d_{r_1}$. As before, from $\alpha d_{r_1} \neq 0$ and $a_{r_1+1,r_1} \in \{0, 1-\alpha\}$, we obtain $\frac{1}{\alpha} = d_{r_1} + 1$.

Finally, if $\boldsymbol{x}$ and $\boldsymbol{y}$ occur one time each and $\boldsymbol{z}$ occurs at least two times, in a very similar way we arrive at $\alpha d_{r_3} + 1 = 1 - \alpha$.

This completes the proof. □

2.2 Main Result

Our main result reads as follows.

Theorem 2.1 Let G be a simple graph with adjacency matrix $A(G)$, diagonal matrix of vertex degrees $D(G)$ and $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, with $\alpha \in [0,1]$. The following statements hold true.

(a) $\operatorname{rank}(A_0) = 2$ if and only if $G \cong K_{n_1,n_2} \cup (n-n_1-n_2)K_1$.

(b) $\operatorname{rank}(A_1) = 2$ if and only if $G \cong K_2 \cup (n-2)K_1$.

(c) For $\alpha \in (0,1)$, if $\operatorname{rank}(A(G)) = 2$, then $\operatorname{rank}(A_\alpha) = 2$ if and only if G is isomorphic to one of the following graphs:

(i) $K_{1,2} \cup (n-3)K_1$, along with $\alpha = \frac{1}{2}$ or

(ii) $K_2 \cup (n-2)K_1$, along with $\alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

(d) For $\alpha \in (0,1)$, if $\operatorname{rank}(A(G)) \neq 2$, then $\operatorname{rank}(A_\alpha) = 2$ if and only if $G \cong (K_1 \vee 2K_{\frac{1}{\alpha}-1}) \cup (n - \frac{2}{\alpha} + 1)K_1$ or $G \cong (2K_{\frac{1}{\alpha}}) \cup (n - \frac{2}{\alpha})K_1$, where for both graphs $2 \leq \frac{1}{\alpha} \in \mathbb{N}$.

Proof The matrix A_0 coincides with the adjacency matrix A and the item (a) is proved by Constantine[5]. For (b), since $A_1 = D$ we have that $\operatorname{rank}(A_1) = 2$ holds if and only if G has exactly two vertices whose degrees are distinct from 0. Obviously, $G \cong K_2 \cup (n-2)K_1$, and we are done.

(c) Here we set $\alpha \in (0,1)$ and $\operatorname{rank}(A) = 2$. It is evident that $K_{1,2} \cup (n-3)K_1$ and $K_2 \cup (n-2)K_1$ satisfy the assumption on the rank of A. In addition, by direct computation we find $\operatorname{rank}(A_{1/2}(K_{1,2} \cup (n-3)K_1)) = 2$ and $\operatorname{rank}(A_\alpha(K_2 \cup (n-2)K_1)) = 2$, for any $\alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Thus, one implication follows. The opposite one is more complicated.

Under the fixed assumptions on α and the rank of A, suppose that G is a graph with $\operatorname{rank}(A_\alpha) = 2$. Item (a) of this theorem gives all graphs whose adjacency matrix has rank 2, and therefore G has the form $K_{n_1,n_2} \cup (n-n_1-n_2)K_1$. It remains to determine the parameters n_1, n_2 for which $\operatorname{rank}(A_\alpha) = 2$ holds true. Without loss of generality, we may assume that $n_1 \leq n_2$.

Case c.1: $n_1 = 1$.

For $n_2 = 1$, we have $G \cong K_2 \cup (n-2)K_1$. We compute $\operatorname{rank}(A_\alpha) = 2$ if $\alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. For $\alpha = \frac{1}{2}$, we obtain

$$A_\alpha = \begin{bmatrix} \frac{1}{2}J_{2\times 2} & O \\ O & O \end{bmatrix}$$

along with $\operatorname{rank}(A_\alpha) = 1$.

For $n_2 = 2$, we have $G \cong K_{1,2} \cup (n-3)K_1$, with

$$A_\alpha = \left[\begin{array}{ccc|c} 2\alpha & 1-\alpha & 1-\alpha & \\ 1-\alpha & \alpha & 0 & O \\ 1-\alpha & 0 & \alpha & \\ \hline & O & & O \end{array}\right].$$

From $\operatorname{rank}(A_\alpha) = 2$, we have

$$\det \begin{bmatrix} 2\alpha & 1-\alpha & 1-\alpha \\ 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & \alpha \end{bmatrix} = 2\alpha(2\alpha - 1) = 0.$$

Hence, $\alpha = \frac{1}{2}$.

For $n_2 \geq 3$, we have

$$A_\alpha = \left[\begin{array}{cccc|c} 3\alpha & 1-\alpha & 1-\alpha & 1-\alpha & \\ 1-\alpha & \alpha & 0 & 0 & \\ 1-\alpha & 0 & \alpha & 0 & * \\ 1-\alpha & 0 & 0 & \alpha & \\ \hline & & * & & * \end{array}\right],$$

where $\alpha \in (0, 1)$. It is obvious that $\operatorname{rank}(A_\alpha) \geq 3$ holds for every $\alpha \in (0, 1)$, i.e., there are no solutions for $n_2 \geq 3$.

Case c.2: $n_1 = 2$. (We will see that this and the next case do not produce any solution.)

For $n_2 = 2$, we have

$$A_\alpha = \left[\begin{array}{cccc|c} 2\alpha & 0 & 1-\alpha & 1-\alpha & \\ 0 & 2\alpha & 1-\alpha & 1-\alpha & \\ 1-\alpha & 1-\alpha & 2\alpha & 0 & O \\ 1-\alpha & 1-\alpha & 0 & 2\alpha & \\ \hline & & O & & O \end{array}\right] = \left[\begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \hline O \end{array}\right],$$

where β_i $(1 \leq i \leq 4)$ is the corresponding row of A_α. Taking into account that $\operatorname{rank}(A_\alpha) = 2$, we have $\beta_3 = u\beta_1 + v\beta_2$. Thus $(u+v)(1-\alpha) = 2\alpha$ and $(u+v)(1-\alpha) = 0$, which is impossible under the assumption that $\alpha \in (0, 1)$.

For $n_2 \geq 3$, we have

$$A_\alpha = \left[\begin{array}{cc|ccc|c} \alpha d_1 & 0 & 1-\alpha & 1-\alpha & 1-\alpha & * \\ 0 & \alpha d_2 & 1-\alpha & 1-\alpha & 1-\alpha & \\ \hline 1-\alpha & 1-\alpha & \alpha d_3 & 0 & 0 & \\ 1-\alpha & 1-\alpha & 0 & \alpha d_4 & 0 & O \\ 1-\alpha & 1-\alpha & 0 & 0 & \alpha d_5 & \\ \hline & * & & O & & * \end{array}\right],$$

where $d_1 = d_2 = n_2$, $d_3 = d_4 = d_5 = 2$. Taking into account that $\alpha \neq 0$, we arrive at $\text{rank}(A_\alpha) \geq 3$.

Case c.3: $n_1 \geq 3$.

Here we have

$$A_\alpha = \left[\begin{array}{ccc|c} \alpha d_1 & 0 & 0 & \\ 0 & \alpha d_2 & 0 & * \\ 0 & 0 & \alpha d_3 & \\ \hline & * & & * \end{array}\right],$$

where $d_1 = d_2 = d_3 = n_2 \geq 3$ and, as before, $\alpha \neq 0$. This forces $\text{rank}(A_\alpha) \geq 3$. This ends the proof of item (c).

(d) Here we set $\alpha \in (0,1)$ and $\text{rank}(A) \neq 2$. It is easy to see that all graphs of the form $(K_1 \vee 2K_{\frac{1}{\alpha}-1}) \cup (n - \frac{2}{\alpha} + 1)K_1$ or $(2K_{\frac{1}{\alpha}}) \cup (n - \frac{2}{\alpha})K_1$, with $2 \leq \frac{1}{\alpha} \in \mathbb{N}$, satisfy the assumption on the rank of A and have matrix A_α of rank 2.

Conversely, suppose that G is a graph with $\text{rank}(A_\alpha) = 2$. Due to the identity (2.1) and Proposition 2.3, A_α has the form

$$A_\alpha = \left[\begin{array}{cccc} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r_1-1,1} & a_{r_1-1,2} & \cdots & a_{r_1-1,n} \\ a_{r_1,1} & a_{r_1,2} & \cdots & a_{r_1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r_2-1,1} & a_{r_2-1,2} & \cdots & a_{r_2-1,n} \\ a_{r_2,1} & a_{r_2,2} & \cdots & a_{r_2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r_3-1,1} & a_{r_3-1,2} & \cdots & a_{r_3-1,n} \\ \hline & O & & \end{array}\right] = \left[\begin{array}{c} x \\ \vdots \\ x \\ y \\ \vdots \\ y \\ z \\ \vdots \\ z \\ \hline O \end{array}\right] = \left[\begin{array}{ccc|c} F_{11} & F_{12} & F_{13} & \\ F_{21} & F_{22} & F_{23} & O \\ F_{31} & F_{32} & F_{33} & \\ \hline & O & & O \end{array}\right], \tag{2.10}$$

and contains either two or three non-equal non-zero rows. We consider these possibilities separately.

Case d.1: A_α has three distinct non-zero rows.

Then $F_{33} \neq 0$ by (2.10).

If $a_{1,r_1} = a_{1,r_2} = a_{r_1,r_2} = 1 - \alpha$ then $F_{21}^\intercal = F_{12} = (1-\alpha)J_{r_1-1\times r_2-r_1}$, $F_{31}^\intercal = F_{13} = (1-\alpha)J_{r_1-1\times r_3-r_2}$ and $F_{32}^\intercal = F_{23} = (1-\alpha)J_{r_2-r_1\times r_3-r_2}$ by (2.5) and (2.7). Thus, G is a complete graph with additional isolated vertices. Let d be the degree of every non-isolated vertex of G. We compute

$$\det \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \big(\alpha d + d(1-\alpha)\big)\big(\alpha d - (1-\alpha)\big)^d.$$

If $\alpha d = 1 - \alpha$, then $\text{rank}(A_\alpha) = 1$. If $\alpha d \neq 1 - \alpha$, then the previous determinant is not equal to 0, thereby $\text{rank}(A_\alpha) \geq 3$. Consequently, at least one of a_{1,r_1}, a_{1,r_2} and a_{r_1,r_2} is 0. If so then, by Proposition 2.2, some of a_{1,r_1}, a_{1,r_2} and a_{r_1,r_2} is 0. Accordingly, since the rows in A_α

are ordered non-increasingly by their sums, we have

$$A_\alpha = \left[\begin{array}{ccc|c} F_{11} & (1-\alpha)J & (1-\alpha)J & \\ (1-\alpha)J & F_{22} & O & O \\ (1-\alpha)J & O & F_{33} & \\ \hline & O & & O \end{array}\right].$$

Suppose that $\boldsymbol{x}$ occurs a times, $\boldsymbol{y}$ occurs b times and $\boldsymbol{z}$ occurs c times (with $a, b, c \geq 1$). Taking into account the identities (2.3)–(2.4) and (2.6), we obtain

$$\begin{cases} d_1 = a + b + c - 1, \\ d_{r_1} = a + b - 1, \\ d_{r_2} = a + c - 1. \end{cases} \tag{2.11}$$

Since $d_1 > d_{r_1} \geq d_{r_2} > 0$, we have $b \geq c$. We proceed with two subcases depending on c.

Subcase d.1.1: $c = 1$. (This subcase will not produce any solution.)

If $(a, b, c) = (1, 1, 1)$, then $(d_1, d_{r_1}, d_{r_2}) = (2, 1, 1)$ by (2.11). This leads to $G \cong K_{1,2} \cup (n-3)K_1$, along with $\operatorname{rank}(A) = 2$, a contradiction.

If $(a, b, c) = (a, 1, 1)$, with $a \geq 2$, then $d_{r_1} = d_{r_2} = a$ and $d_1 = a + 1$ by (2.11). The first a rows of A_α coincide with $\boldsymbol{x}$, which yields $\alpha d_1 = a_{1,1} = a_{2,1} = \cdots = a_{a,1}$. Since $a_{2,1} = \cdots = a_{a,1} \in \{0, 1-\alpha\}$, we have $a_{1,1} = 1 - \alpha$, while since $a_{1,1} = \alpha d_1 = \alpha(a+1)$, we have $a = \frac{1}{\alpha} - 2$. Thus there exists a submatrix H such that

$$\det(H) = \det \begin{bmatrix} a_{1,1} & a_{1,r_1} & a_{1,r_2} \\ a_{r_1,1} & a_{r_1,r_1} & a_{r_1,r_2} \\ a_{r_2,1} & a_{r_2,r_1} & a_{r_2,r_2} \end{bmatrix} = \det \begin{bmatrix} 1-\alpha & 1-\alpha & 1-\alpha \\ 1-\alpha & 1-2\alpha & 0 \\ 1-\alpha & 0 & 1-2\alpha \end{bmatrix} = -(2\alpha-1)(\alpha-1).$$

Since $\operatorname{rank}(A_\alpha) = 2$, there must be $\det(H) = 0$, which implies $\alpha = \frac{1}{2}$, but this leads to $a = 0$, a contradiction.

If $(a, b, c) = (1, b, 1)$, with $b \geq 2$, then $d_1 = b + 1, d_{r_1} = b, d_{r_2} = 1$ by (2.11). Thus the rows starting from the r_1th and ending with the $(r_1 + b - 1)$th coincide with $\boldsymbol{y}$, and so $\alpha d_{r_1} = a_{r_1,r_1} = a_{r_1+1,r_1} = \cdots = a_{r_1+b-1,r_1}$. As before, since $a_{r_1+1,r_1} = \cdots = a_{r_1+b-1,r_1} \in \{0, 1-\alpha\}$, we have $a_{r_1,r_1} = 1 - \alpha$, while since $a_{r_1,r_1} = \alpha d_{r_1} = \alpha b$, we also have $b = \frac{1}{\alpha} - 1$. Thus there exists a submatrix denoted again by H such that

$$\det(H) = \det \begin{bmatrix} a_{1,1} & a_{1,r_1} & a_{1,r_2} \\ a_{r_1,1} & a_{r_1,r_1} & a_{r_1,r_2} \\ a_{r_2,1} & a_{r_2,r_1} & a_{r_2,r_2} \end{bmatrix} = \det \begin{bmatrix} 1 & 1-\alpha & 1-\alpha \\ 1-\alpha & 1-\alpha & 0 \\ 1-\alpha & 0 & \alpha \end{bmatrix} = -(2\alpha-1)(\alpha-1).$$

Again, there must be $\det(H) = 0$, i.e., $\alpha = \frac{1}{2}$, but this implies $b = 1$, a contradiction.

If $(a, b, c) = (a, b, 1)$, with $a \geq 2$, $b \geq 2$, then the first a rows coincide with $\boldsymbol{x}$ and they are followed by b rows that coincide with $\boldsymbol{y}$. Thus $\alpha d_1 = a_{1,1} = a_{2,1} \in \{0, 1-\alpha\}$ and $\alpha d_{r_1} = a_{r_1,r_1} = a_{r_1+1,r_1} \in \{0, 1-\alpha\}$. Since $a_{1,1} \neq 0$ and $a_{r_1,r_1} \neq 0$, we have $d_1 = d_{r_1} = \frac{1}{\alpha} - 1$, which contradicts the inequality $d_1 > d_{r_1}$ established below the equalities (2.11).

Subcase d.1.2: $c \geq 2$.

If $a \geq 2$, then $\alpha d_1 = \alpha d_{r_1} = \alpha d_{r_2} = 1-\alpha$, i.e., $d_1 = d_{r_1} = \frac{1}{\alpha} - 1$ which contradicts $d_1 > d_{r_1}$. Thus, $a = 1$, i.e.,

$$A_\alpha = \left[\begin{array}{ccc|c} 2(1-\alpha) & (1-\alpha)J_{1\times k} & (1-\alpha)J_{1\times k} & \\ (1-\alpha)J_{k\times 1} & (1-\alpha)J_{s\times k} & O & O \\ (1-\alpha)J_{k\times 1} & O & (1-\alpha)J_{s\times k} & \\ \hline & O & & O \end{array}\right],$$

where $k = \frac{1}{\alpha} - 1$. Thus $G \cong (K_1 \vee 2K_{\frac{1}{\alpha}-1}) \cup (n - \frac{2}{\alpha} + 1)K_1$, with $2 \leq \frac{1}{\alpha} \in \mathbb{N}$.

Case d.2: A_α has only two distinct non-zero rows.

From (2.10), we have

$$A_\alpha = \left[\begin{array}{cc|c} F_{11} & F_{12} & O \\ F_{21} & F_{22} & \\ \hline & O & O \end{array}\right]. \tag{2.12}$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ occur one time each, then $G \cong K_2 \cup (n-2)K_1$, which leads to $\mathrm{rank}(A) = 2$.

If one of $\boldsymbol{x}, \boldsymbol{y}$ occurs one time and the other occurs at least two times, and also $F_{12} = O$ and $F_{21} = O$, then A_α contains a row with exactly one non-zero entry which is impossible under the assumption that $\alpha \in (0, 1)$.

If one of $\boldsymbol{x}, \boldsymbol{y}$ occurs one time and the other occurs at least two times along with $F_{12} = (1-\alpha)J$ and $F_{21} = (1-\alpha)J$, by taking into account (2.3)–(2.4) and (2.12) we find that G is a complete graph with additional isolated vertices. If the vertex degree in the non-trivial component of G is $\frac{1}{\alpha} - 1$, then $\mathrm{rank}(A_\alpha) = 1$. For any other vertex degree we compute $\mathrm{rank}(A_\alpha) \geq 3$.

If both $\boldsymbol{x}, \boldsymbol{y}$ occur at least two times each, and also $F_{12} = O$, $F_{21} = O$, then

$$A_\alpha = \left[\begin{array}{cc|c} (1-\alpha)J & O & O \\ O & (1-\alpha)J & \\ \hline & O & O \end{array}\right],$$

where $J = J_{\frac{1}{\alpha}\times\frac{1}{\alpha}}$. This leads to $G \cong (2K_{\frac{1}{\alpha}}) \cup (n - \frac{2}{\alpha})K_1$, with $2 \leq \frac{1}{\alpha} \in \mathbb{N}$.

Finally, if, under the latter assumption on $\boldsymbol{x}, \boldsymbol{y}$, we have $F_{12} = (1-\alpha)J$ and $F_{21} = (1-\alpha)J$ then $\boldsymbol{x} = \boldsymbol{y}$, which is impossible.

The proof is complete. □

The following result follows from $\alpha = \frac{1}{2}$ in Theorem 2.1.

Theorem 2.2 Let G be a graph with order n. Then $\mathrm{rank}_Q(G) = 2$ iff

$$G \in \{K_{1,2} \cup (n-3)K_1, (K_1 \vee 2K_1) \cup (n-3)K_1, 2K_2 \cup (n-4)K_1\}.$$

3 Conclusion

In this paper, we systematically summarize various graphs with small ranks, including mixed graphs, oriented graphs, complex unit gain graphs and sign graphs. Moreover, we determine all graphs with A_α-rank 2, which generalizes the graphs with A-rank 2. As a corollary, we characterize the graphs with Q-rank 2. Naturally, it seems to be an interesting problem that identifying the graphs with A_α-ranks 3, 4 and 5.

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References

[1] Chang, G.J., Huang, L.H. and Yeh, H.-G., A characterization of graphs with rank 4, *Linear Algebra Appl.*, 2011, 434(8): 1793–1798.

[2] Chang, G.J., Huang, L.H. and Yeh, H.-G., A characterization of graphs with rank 5, *Linear Algebra Appl.*, 2012, 436(11): 4241–4250.

[3] Cheng, B. and Liu, B.L., On the nullity of tricyclic graphs, *Linear Algebra Appl.*, 2011, 434(8): 1799–1810.

[4] Collatz, L. and Sinogowitz, U., Spektren endlicher Grafen, *Abh. Math. Sem. Univ. Hamburg*, 1957, 21: 63–77 (in German).

[5] Constantine, G., Lower bounds on the spectra of symmetric matrices with nonnegative entries, *Linear Algebra Appl.*, 1985, 65: 171–178.

[6] Cvetković, D.M., Doob, M. and Sachs, H., Spectra of Graphs: Theory and Application, New York: Academic Press, 1980.

[7] Cvetković, D.M. and Gutman, I.M., The algebraic multiplicity of the number zero in the spectrum of a bipartite graph, *Mat. Vesnik*, 1972, 9(24): 141–150.

[8] Cvetković, D.M., Rowlinson, P. and Simić, S., An Introduction to the Theory of Graph Spectra, London Mathematical Society Student Texts, Vol. 75, Cambridge: Cambridge University Press, 2010.

[9] Van Dam, E.R. and Haemers, W.H., Which graphs are determined by their spectrum? *Linear Algebra Appl.*, 2003, 373: 241–272.

[10] Das, K.C., Lin, H.Q. and Guo, J.M., Distance signless Laplacian eigenvalues of graphs, *Front. Math. China*, 2019, 14(4): 693–713.

[11] Fan, Y.Z. and Qian, K.S., On the nullity of bipartite graphs, *Linear Algebra Appl.*, 2009, 430(11/12): 2943–2949.

[12] Godsil, C.D., Inverses of trees, *Combinatorica*, 1985, 5(1): 33–39.

[13] Guo, J.M., Yan, W.G. and Yeh, Y.-N., On the nullity and the matching number of unicyclic graphs, *Linear Algebra Appl.*, 2009, 431(8): 1293–1301.

[14] Gutman, I. and Rouvray, D.H., An approximate topological formula for the homo-lumo separation in alternant hydrocarbons, *Chem. Phys. Lett.*, 1979, 62(2): 384–388.

[15] Gutman, I. and Sciriha, I., On the nullity of line graphs of trees, *Discrete Math.*, 2001, 232(1/2/3): 35–45.

[16] Haemers, W.H. and Omidi, G.R., Universal adjacency matrices with two eigenvalues, *Linear Algebra Appl.*, 2011, 435(10): 2520–2529.

[17] Hu, S.B., Tan, X.Z. and Liu, B.L., On the nullity of bicyclic graphs, *Linear Algebra Appl.*, 2008, 429(7): 1387–1391.

[18] Huang, H., Induced subgraphs of hypercubes and a proof of the sensitivity conjecture, *Ann. of Math. (2)*, 2019, 190(3): 949–955.

[19] IMA-ISU research group on minimum rank, Minimum rank of skew-symmetric matrices described by a graph, *Linear Algebra Appl.*, 2010, 432(10): 2457–2472.

[20] Jia, H.C. and Shiu, W.C., Distance signless Laplacian spectrum of a graph, *Front. Math. China*, 2022, 17(4): 653–672.

[21] Jiang, Z.L., Tidor, J., Yao, Y., Zhang, S.T. and Zhao, Y.F., Equiangular lines with a fixed angle, *Ann. of Math. (2)*, 2021, 194(3): 729–743.

[22] Kirchhoff, G., Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird, *Annalen der Physik und Chemie*, 1847, 148(12): 497–508 (in German).

[23] Li, S.C., On the nullity of graphs with pendent vertices, *Linear Algebra Appl.*, 2008, 429(7): 1619–1628.

[24] Li, X.L. and Yu, G.H., The skew-rank of oriented graphs, *Sci. Sin. Math.*, 2015, 45(1): 93–104 (in Chinese).

[25] Liu, B.L., Huang, Y.F. and Chen, S.Y., On the characterization of graphs with pendent vertices and given nullity, *Electron. J. Linear Algebra*, 2009, 18: 719–734.

[26] Lu, Y., Wang, L.G. and Xiao, P., Complex unit gain bicyclic graphs with rank 2, 3 or 4, *Linear Algebra Appl.*, 2017, 523: 169–186.

[27] Lu, Y., Wang, L.G. and Zhou, Q.N., Bicyclic oriented graphs with skew-rank 6, *Appl. Math. Comput.*, 2015, 270: 899–908.

[28] Lu, Y., Wang, L.G. and Zhou, Q.N., The rank of a signed graph in terms of the rank of its underlying graph, *Linear Algebra Appl.*, 2018, 538: 166–186.

[29] McLeman, C. and McNicholas, E., Graph invertibility, *Graphs Combin.*, 2014, 30(4): 977–1002.

[30] Mohar, B., Hermitian adjacency spectrum and switching equivalence of mixed graphs, *Linear Algebra Appl.*, 2016, 489: 324–340.

[31] Nikiforov, V., Merging the A- and Q-spectral theories, *Appl. Anal. Discrete Math.* 2017, 11(1): 81–107.

[32] Qu, H. and Yu, G.H., Bicyclic oriented graphs with skew-rank 2 or 4, *Appl. Math. Comput.*, 2015, 258: 182–191.

[33] Qu, H., Yu, G.H. and Feng, L.H., More on the minimum skew-rank of graphs, *Oper. Matrices*, 2015, 9(2): 311–324.

[34] Sciriha, I., On the rank of graphs, In: Combinatorics, Graph Theory, and Algorithms, Vol. I, II, Kalamazoo, MI: New Issues Press, 1999, 769–778.

[35] Shader, B. and So, W., Skew spectra of oriented graphs, *Electron. J. Combin.*, 2009, 16(1): Note 32, 6 pp.

[36] Simion, R. and Cao, D.S., Solution to a problem of C. D. Godsil regarding bipartite graphs with unique perfect matching, *Combinatorica*, 1989, 9(1): 85–89.

[37] Tang, X.Z. and Liu, B.L., On the nullity of unicyclic graphs, *Linear Algebra Appl.*, 2005, 408: 212–220.

[38] Trinajstić, N., Chemical Graph Theory, 2nd Ed., Mathematical Chemistry Series, Boca Raton, FL: CRC Press, 1992.

[39] Wang, Y., Yuan, B.J., Li, S.D. and Wang, C.J., Mixed graphs with H-rank 3, *Linear Algebra Appl.*, 2017, 524: 22–34.

[40] Xu, F., Zhou, Q., Wong, D. and Tian, F.L., Complex unit gain graphs of rank 2, *Linear Algebra Appl.*, 2020, 597: 155–169.

[41] Yang, J.L., Wang, L.G. and Yang, X.W., Some mixed graphs with H-rank 4, 6 or 8, *J. Comb. Optim.*, 2021, 41(3): 678–693.

[42] Yang, Y.J. and Ye, D., Inverses of bipartite graphs, *Combinatorica*, 2018, 38(5): 1251–1263.

[43] Ye, D., Yang, Y.J., Mandal, B. and Klein, D.J., Graph invertibility and median eigenvalues, *Linear Algebra Appl.*, 2017, 513: 304–323.

[44] Yeh, Y.-N., Gutman, I. and Fu, C.-M., Graph transformations which preserve the multiplicity of an eigenvalue, *Discrete Appl. Math.*, 1996, 67(1/2/3): 221–228.

[45] Yu, G.H., Qu, H. and Tu, J.H., Inertia of complex unit gain graphs, *Appl. Math. Comput.*, 2015, 265: 619–629.

[46] Zhang, Y.S., Xu, F. and Wong, D., Characterization of oriented graphs of rank 2, *Linear Algebra Appl.*, 2019, 579: 136–147.

具有较小秩的图：新旧结论

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摘要： 一个图 G 的 M- 秩是其图矩阵 $M(G)$ 的秩. 本文总结了具有相对较小的 M- 秩的图的结果, 并提出了可进一步研究的若干问题. 此外, 对于实数 $\alpha \in [0,1]$, 本文确定了所有 A_α- 秩为 2 的图类, 其中 $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, $A(G)$ 和 $D(G)$ 分别是图 G 的邻接矩阵和度矩阵.

关键词： 邻接矩阵; 度矩阵; 矩阵的凸线性组合; 矩阵的秩